The National Strategies
Secondary

Teaching mental mathematics from level 5

Algebra

department for
children, schools and families

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Algebra

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Contents

00692-2009PDF-EN-01

Introduction

What is mental mathematics?

Almost all of mathematics could be described as 'mental' in the sense that engaging in a mathematical task involves thinking. Thus every mathematical problem a pupil tackles must involve several stages of mental mathematics. Pupils actively involved in mental mathematics might be engaged in any combination of:

interpreting visualising analysing synthesising explaining hypothesising inferring deducing judging justifying making decisions

These ideas are prevalent throughout mathematics and underpin mathematical processes and applications.

If the definition is so wide ranging, how have we produced a few brief booklets with this title? The answer is that we have been very selective! The 'mental mathematics' supported through the teaching approaches described in these booklets is aimed at a subset of mental mathematics in its broadest sense. We have chosen a few key areas likely to influence pupils' progress beyond level 5. These selections have been informed by recent annual standards reports from the Qualifications and Curriculum Development Agency (QCDA) and the experience of teachers and consultants. The initial ideas have also been supported by classroom trials.

How do I help pupils to improve the way they process mathematics mentally?

Individual pupils will be at different stages but all pupils develop some strategies for processing mathematical ideas in their heads. Many of the activities suggested in these booklets increase the opportunities for pupils to learn from one another by setting them to work collaboratively on tasks that require them to talk. Often pupils develop and enhance their understanding after they have tried to express their thoughts aloud. It is as if they hear and recognise inconsistencies when they have to verbalise their ideas.

Equally, new connections can be made in a pupil's 'mental map' when, at a crucial thinking point, they hear a different slant on an idea. A more discursive way of working often allows pupils to express a deeper and richer level of understanding of underlying concepts that may otherwise not be available to them. In this way pupils may:

- reach a greater facility level with pre-learned skills, for example, becoming able to solve simple linear equations mentally
- achieve a leap in understanding that helps to complete 'the big picture', for example, seeing how the elements of a function describing the position-to-term relationship in a sequence are generated from elements in the context of the sequence itself.

The activities are designed to engage pupils in group work and mathematical talk.

Is mental mathematics just about the starter to the lesson?

Developing mental processes is not simply about keeping some skills sharp and automating processes through practice. The activities described in this booklet support the main part of the lesson. Developing a mental map of a mathematical concept helps pupils to begin to see connections and use them to help solve problems. Developing the ability to think clearly in this way takes time. Once in place, some aspects of mental mathematics can be incorporated into the beginning of lessons as a stimulating precursor to developing that topic further.

The activities are intended to support the main part of the lesson.

Is mental mathematics just about performance in mental tests?

Using these materials will help pupils to perform more successfully in tests, but the aim is more ambitious than that. Developing more effective mental strategies for processing mathematical ideas will impact on pupils' progress in mathematics and their confidence to apply their skills to solve problems.

Secondary teachers recognise the importance of pupils' mathematical thinking and application, but few have a range of strategies to support its development. The expectations described, and the activities suggested in the accompanying mental mathematics resources, aim to create a level of challenge that will take pupils further in their thinking and understanding. These materials should provide the chance for pupils to interact in such a way that they learn from each other's thinking, successes and misconceptions and thereby become increasingly confident and independent learners.

Pupils need to transfer mathematics confidently and apply it whenever they need to use it. This needs to be taught. Most commonly, pupils will use mental mathematics in solving problems as they occur in their lives, in other areas of their studies and as they prepare for the world of work. To support pupils in doing this, teachers will frequently need to set both large and small mathematical problems in real, purposeful and relevant contexts. Pupils will need to solve increasingly complex and unfamiliar problems using mathematics, apply more demanding mathematical procedures during their analysis and do so with increasing independence. These materials support teachers in planning a structured and progressive approach to do this. If learning is planned with mental mathematics as a significant element, pupils will develop increasing confidence in applying mathematics.

Improving mental mathematics will improve pupils' confidence to apply what they know.

Can mental mathematics involve paper and pencil?

Mathematical thinking involves drawing on our understanding of a particular concept, making connections with related concepts and previous problems and selecting a strategy accordingly. Some of these decisions and the subsequent steps in achieving solutions are committed to paper and some are not. When solving problems, some of the recording becomes part of the final solution and some will be disposable jottings.

Many of the activities involve some recording to stimulate thinking and talking. Where possible, such recording should be made on large sheets of paper or whiteboards. This enables pupils, whether working as a whole class or in pairs or small groups, to share ideas. Such sharing allows them to see how other pupils are interpreting and understanding some of the big mathematical ideas. Other resources such as diagrams, graphs, cards, graphing calculators and ICT software are used in the activities. Many of these are reusable and, once developed in the main part of a lesson, can be used more briefly as a starter on other occasions.

Progress may not appear as written output. Gather evidence during group work by taking notes as you listen in on group discussions. Feed these notes into the plenary and use them in future planning.

The materials

Each attainment target in mathematics is addressed through its own booklet, divided into separate topic areas. For each topic, there is a progression chart that illustrates expectations for mental processes, broadly from level 5 to level 8. Mathematical ideas and pupils' learning are not simple to describe, nor do they develop in a linear fashion. These are not rigid hierarchies and the degree of demand will be influenced by the context in which they occur and, particularly for the number topics, by the specific numbers involved. For this reason ideas from one chart have to interconnect with those in another. The aim is that the charts will help teachers to adjust the pitch of the activities that are described on subsequent pages.

There are many National Strategies materials which reinforce and extend these ideas but, to ensure that these booklets are straightforward and easy for teachers to use, cross-referencing has been kept to a minimum. The most frequent referencing throughout the booklet is to the *Supplement of examples*, which is now connected to the Framework for Secondary Mathematics (**www.standards.dcsf.gov.uk/nationalstrategies**). The page numbers of the original supplement have been retained and the examples can be downloaded as a complete document or in smaller sets from the related learning objectives.

Teaching mental mathematics from level 5: Algebra

The topics covered in this chapter are:

- algebraic conventions
- solving linear equations
- sequences
- functions and graphs.

These are selected from the algebra section of the learning objectives on the *Framework for secondary mathematics*. They also include some of the aspects of algebra that have been reported as having implications for teaching and learning from the Key Stage 3 tests. For example, to help pupils improve their performance, teachers should:

- give pupils more experience of forming equations and help them to understand the use of inverse operations when solving or rearranging equations
- help pupils to improve their understanding of the meaning of coefficients and symbols, taking account of common misconceptions
- provide opportunities to practise transformations of algebraic expressions for pupils at higher levels.

Some of the topics have been selected because, although they are fundamental ideas, it can be difficult to map or identify pupils' acquisition of them. For example, pupils gradually gain thorough understanding of the meaning of symbols through various different experiences. Gaps in their understanding can undermine later stages of pupils' learning.

The tasks described in this chapter aim to address some typical algebraic misconceptions. They are designed to engage pupils with algebra through discussion and collaborative work. The tasks may easily be adapted to adjust the challenge and keep pupils on the edge of their thinking.

In the following sections two strategies are used repeatedly and are worth general consideration.

- **Classifying** is a task well suited to the thinking processes that everyone uses naturally to organise information and ideas. A typical classification task may involve a card sort. Pupils work together to sort cards into groups with common characteristics that establish criteria for classification. Being asked to consider and justify their criteria helps pupils to develop their skills and understanding. The key part of designing a good classification task is the initial choice of cards that will provide a sufficiently high challenge. A common mistake when running a classification task is to intervene too soon and over-direct the pupils.
- **Matching** different forms of representation often involves carefully-selected cards and a common lesson design. In this instance pupils are asked to match cards that are equivalent in some way. This kind of activity can give pupils important mental images, at the same time offering the chance to confront misconceptions.

Algebraic conventions

Pupils need to be as familiar with the conventions of algebra as they are with those of arithmetic. Algebraic conventions should become a routine part of algebraic thinking, allowing greater access to more challenging problems. It is a common error to deal with these conventions rather too quickly. How pupils understand and manipulate algebraic forms is determined by their mental processing of the meaning of the symbols and the extent to which they can distinguish one algebraic form from another.

We are seeking to develop a mental facility to recognise which type of algebraic form is presented or needs to be constructed as part of a problem. Some time spent on this stage of the process can reduce misconceptions when later problems become quite complex.

Recognise and explain the use of symbols

Explicitly model and explain the correct vocabulary. For example, in the equation $p + 7 = 20$ the letter p represents a particular unknown number, whereas in $p + q = 20$, p and q can each take on any one of a set of different values and can therefore be called variables. Equations, formulae and functions can describe relationships between variables. In a function such as $q = 3p + 5$ we would say that $3p$ was a variable term, whereas 5 is a constant term. Be precise and explicit in using this vocabulary and expect similar usage by pupils.

$3x + 5 = 11$	Represent an unknown value in equations with a unique solution
$p + q = 20$	Represent unknown values in equations with a set of solutions
$2l + 2b = p$	Represent variables in formulae
$y = \frac{x}{2} - 7$	Represent variables in functions

Identify equivalent terms and expressions

It is often the case that pupils do not realise when an equation or expression has been changed, or when it looks different but is in fact still the same. The ability to recognise and preserve equivalent forms is a very important skill in algebraic manipulation and one in which pupils need practice. One way of approaching this is to start with simple cases and generate more complex, but equivalent, forms. This can then be supported by tasks involving matching and classifying.

$2x + x + 5$	simple chains of operations
$ax + 5$	some with unknown coefficients
$7(x + 2)$	brackets (linear)
$(x + 2)(x + 5)$	brackets (quadratic)
$x^3 \times x$	positive indices

Identify types and forms of formulae

This will build on the understanding of equivalence and will rely on knowledge of commutativity and inverse. Encourage pupils to see general structure in formulae by identifying small collections of terms as ‘objects’. These objects can then be considered as replacing the numbers in ‘families of facts’, such as $3 + 5 = 8$, $5 + 3 = 8$, $3 = 8 - 5$, $5 = 8 - 3$. The equations are then more easily manipulated mentally.

To develop pupils’ understanding of the dimensions of a formula, make explicit connections between the structure of the formula and its meaning. Consider the units associated with each variable and how these build up, term by term. Involve pupils in generating and explaining non-standard formulae, for example, for composite shapes.

$\frac{a}{b} = l$, $a = l \times b$	equivalence of formulae
$a = l \times b$, $2l + 2b = p$	dimensions of a formula

The *Framework for secondary mathematics* supplement of examples, pages 112 to 117, provides contexts in which pupils should develop mental processes in algebraic conventions.

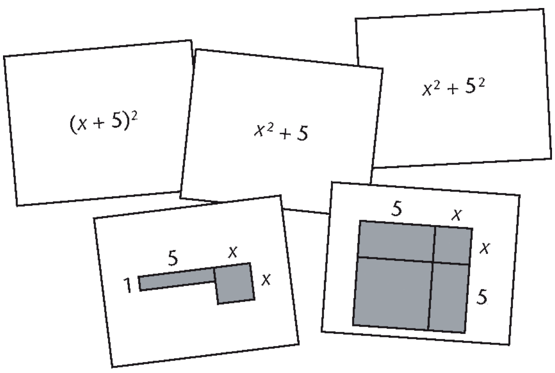
Activities

Classifying cards will help to build pupils’ understanding of the connections within different algebraic forms. A set of cards could include:

- expressions, some equivalent, some not
- equations with unique solutions, some with sets of solutions, some formulae and some functions
- formulae of different dimensions (linear, area, volume, compound).

For an illustration of the types of cards see the *Framework for secondary mathematics* supplement of examples, page 116.

Matching different forms of representation is also helpful. For example, pupils could be asked to match a set of cards with algebraic expressions to another set showing area diagrams. The cards could include typical ‘conflict’ cases.



The *Framework for secondary mathematics* supplement of examples, page 116, uses this same area image to develop pupils’ understanding of the expansion of brackets. It builds on the grid method of long multiplication which, in algebra, can provide a strong mental image of the source of each element within an expansion of brackets.

As another example, could these expressions be equivalent to $6 - 8x$?

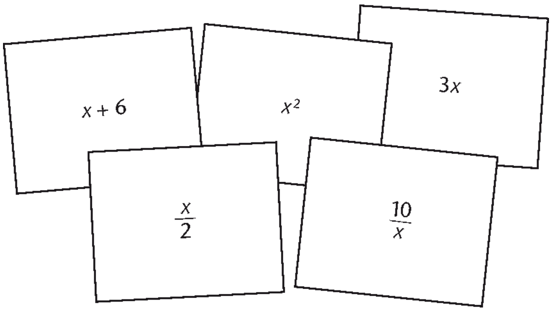
$2 \; \boxed{} \; 4(1 \; \boxed{} \; 2x)$

$10 \; \boxed{} \; 4(1 \; \boxed{} \; 2x)$

The operations in these expressions are hidden by sticky notes. Pupils are asked to apply their mental reasoning, not laboriously expand expressions.

Sequencing expression cards in order of size, by taking *x* to be a given positive integer, could be used as a preliminary task for pupils, before setting more challenging questions such as:

- Which cards would change position if *x* was a given negative integer? (For example, –2 instead of 2.)
- Which cards would change position if the value of *x* was between zero and one?



Considering this point can encourage pupils to check their written strategies, especially where it is important to have an idea of the relative size and nature of the solution.

Pyramid activities give pupils opportunities to construct expressions of various degrees of complexity, as illustrated in the *Framework for secondary mathematics* supplement of examples, page 116.

Keep the task centred on talk and jottings rather than written manipulation and simplification. Focus on mental strategies by asking questions about preceding layers of the pyramid, for example:

- What possible entries in the early stages of this pyramid could result in this expression?
- Could this be one of the entries? Why not?

Clouding the picture, as illustrated on pages 22–24 of this document, could easily be adapted from equations to expressions. Pupils are asked to write an expression 'in as many ways as you can', using a systematic way of 'complicating' the expression. For example, one system for 'complicating' $2x + 8$ might appear as $3x + 8 - x$ and $4x + 8 - 2x$. Ask: 'Are they equivalent? How do you know?'

Look out for pupils explaining the system that they devised to ensure that the expressions were equivalent. Extend to more complex examples such as $5x + 6 + y$ and look for challenging 'complications' such as $\frac{1}{2}(10x + 12 + 2y)$.

Solving linear equations

All pupils should aim to achieve mental agility with the linear form. Their ability to solve algebraic equations at the higher National Curriculum levels can be greatly improved if their responses to these stages of a problem become automatic.

The expectations listed below refer exclusively to the mental facility that will support written solution at level 5 and above.

The progression in the tables below is described in two ways.

- The tables show how the form of the equation is perceived to become more difficult to deal with because of the change in position of the unknown term.
- Within each table the solution involves values that are harder to deal with, such as non-integer or negative integer values.

One-step linear equations

The position of the unknown value is not a source of difficulty if pupils are practised in thinking flexibly about the form of an equation. Encourage them to generalise their pre-knowledge of arithmetical commutativity and inverse in 'families of facts' such as $3 \times 5 = 15$, $5 \times 3 = 15$, $3 = 15 \div 5$, $5 = 15 \div 3$.

One-step linear equations with the unknown in a 'standard' position:	
$x + 4 = 7$	positive integer solutions
$\frac{x}{4} = 6$	
$x - 9 = 34$	
$8x = 56$	
$3x = 5$	non-integer solutions
$x + 14 = 9$	negative integer solutions

One-step linear equations with the unknown perceived to be in a 'harder' position:	
$13 = 8 + x$	positive integer solutions
$\frac{20}{x} = 10$	
$\frac{20}{x} = 3$	non-integer solutions
$13 = 8 - x$	negative integer solutions

Equations involving brackets

One way of solving such equations would involve multiplying out brackets. Another, often simpler, way is to encourage pupils to see the expression in brackets as an object or term. For example, $3(x + 4) = 27$ has the same structure as $3p = 27$, so pupils can mentally move through this step and see that $x + 4 = 9$.

$3(x + 4) = 27$	positive integer solutions
$\frac{(x-5)}{3} = 7$	

Inequalities

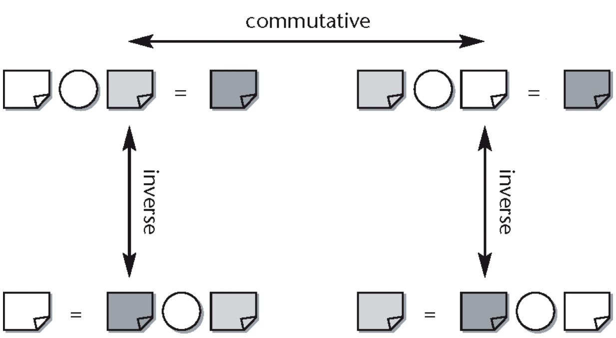
Pupils can better understand an inequality (or 'inequation') if they have the ability to think clearly about the meaning of the 'equals' sign in an equation. Reading 'equals' as 'makes' is a limited interpretation. Use the language of 'equals' meaning 'is the same as' to develop understanding of equations. The consequent step to 'is greater than' can develop form this much more clearly. Consider building understanding by using visual and mental images such as number lines and coordinate axes. Ask pupils to express inequalities both ways round, to develop flexible and thoughtful use of language and symbols.

$5x < 10$	one boundary to solution set, one-step solution
$-4 < 2x < 10$	two boundaries to solution set
$5x + 3 < 10$	two-step solution

The *Framework for secondary mathematics* supplement of examples, pages 122 to 125, provides contexts in which pupils could develop mental processes in algebraic solution.

Activities

Show a **Family of equations** by using a template for the elements and operations in an equation, as shown below. This is illustrated in the example on page 21.



Use colour-coded sticky notes to hide two numbers, one unknown term and the operations (use circles) needed to make a 'family' of equations. Show one of the equations and use pupils' knowledge of arithmetical relationships (families of facts) to generate the three related equations. Encourage pupils to judge which of the four equivalent forms enables them most easily to evaluate the unknown term, using a mental process. Consider changing one element and map through the effect on other equations. *Does the change affect pupils' judgements?*

It is important that pupils remember the structure of these relationships and can recall this as a mental image.

In the above activity, terms can be replaced by expressions, for example, considering $(x + 4)$ as a single term in equations such as:

$3 \times (x + 4) = 27$ or $\frac{(x+4)}{2} = 19$

> Pupils' understanding is strengthened by the regular layout, use of colour, 'hide and reveal' tasks, pupils using their own cards (moving and annotating) and as much talk and shared explanation as possible. These tactics support pupils as they move from the concrete (arithmetic equations) to the abstract (algebraic equations). In particular, pupils have to recognise those operations that are commutative and can be positioned at the top of the template.

The task of **Clouding the picture** is described on pages 22–24 of this document. Pupils are asked to express an equation in as many ways as they can. Use this principle to ask: 'What else can you see?' For example, two equations in **Clouding the picture** are:

$5x + 8 = 11$

$5x + 8 = 3 + 8$

Reading the equals sign as 'is the same as', ask pupils: 'What else can you see?' Read out the second equation and ask if they can see that $5x$ must be the same as 3. Extend to more complex examples such as:

$5x + 8 = x + 11$

$4x + x + 8 = x + 8 + 3$

Pupils see that $4x$ must be the same as 3.

Allow time for pupils to consider this point as it is not obvious to all. Those who can follow the reasoning should be given the chance to explain it.

Extend this activity to inequalities such as:

$3x + 7 < 13$

$3x + 7 < 7 + 6$

Point out that this is not an equals sign. Ask pupils:

- 'What else can you see?'

Clarify that since 7 'is the same as' 7 the inequality (or imbalance) must come about by the value of the $3x$ term being less than 6.

> Be careful with the use of a balance analogy. The idea of matching items that are the same on both sides is more easily extended than the concept of 'taking off' from each side.

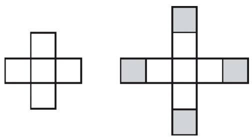
Ask pupils to **Annotate written solutions**. Provide a selection of very detailed written solutions to the same algebraic equation and ask them, first, to choose a method that supports their thinking and then to highlight those stages that they would confidently omit when writing out the solution. Pair pupils and ask them to compare the mental stages and to explain how they process the steps in their heads. Give each pair another equation to solve and ask them to give each other a commentary on what they write and what they think.

Discussing strategies in this way stimulates pupils to think about their thinking (metacognition). Many pupils will refine their own mental processes following collaborative work. This also gives pupils opportunities to discuss the purpose of presenting working and to establish what is required for different audiences. For example, compare introductory notes against revision notes, coursework with a timed examination, a mental with a written test or a calculator and non-calculator examination paper.

Sequences

Sequences can – deceptively – appear to be straightforward. Consequently the early stages of teaching the topic may be rushed so that pupils do not get the chance to compose the 'big picture' for themselves. They need to develop a full understanding of the relationship between the context of a sequence and the ways in which it can be expressed, first in words and later in algebraic terms. It is essential for pupils to understand the term-to-term and position-to-term structures and the relationship between them. Use pupils' knowledge of variables and functions. $T(n)$ and n are not 'shorthand' for words, they are symbols representing variables that can take on specific values (in most examples these are positive integer values). To describe a sequence, we can use either a function that links one term $T(n + 1)$ with the previous term $T(n)$ or a function that links a term $T(n)$ with its position in the sequence n.

Linear sequences: find and describe in words and symbols the rule for the next term and the *n*th term of a sequence



It is important for pupils to use physical patterns, rather than just tables of number sequences, to justify their algebraic forms. Furthermore, exploring different but equivalent forms will help pupils to understand and relate equivalent algebraic expressions and functions.

Pupils will believe that the algebra must be equivalent by relating to the structure but will also be motivated to manipulate the expressions or functions to confirm this equivalence. Be explicit about the use of symbols to represent variables and avoid the temptation to treat the algebra as an abbreviation of the words. Emphasise the meaning of the item they are constructing: for example, a function that relates the number of objects in the shape to its position in the sequence, say $T(n) = 7 + 3n$.

5 , 9 , 13 , 17	Use the context of a sequence to generate the related numerical terms
It goes up in 4s because you add 1 block to each arm of the cross. n 1 2 3 4 T(n) 5 9 13 17 add 4 $T(n + 1) = T(n) + 4$	Notice and describe how the sequence is growing term by term and relate this to the context
For the nth shape there is one block in the middle and then four arms of n blocks. T(n) is the total number of blocks in a shape and n is the position of the shape in the sequence. $T(n) = 1 + 4n$	Notice and describe a general term in the sequence and relate this to the context
There is a horizontal row of blocks, which is one more than twice the term number, then a top leg with n blocks and a bottom leg with n blocks. $T(n) = (2n + 1) + n + n$	Appreciate different forms for the general term and relate each to the context
	Generate different forms for the general term and relate each to the context

Quadratic sequences: find and describe in symbols the rule for the next term and the *n*th term of a sequence

Explore practical contexts for simple cases. Consider using ICT for generating sequences. Use the simple examples linked to difference tables and generalise from these examples.

For examples see the *Framework for secondary mathematics*, pages 151 and 153	Repeat the progression described above

Recognise and describe types of sequence: for example, arithmetical sequences and multiples, triangular numbers, square numbers

Explore familiar spatial patterns that generate multiples, triangular numbers and square numbers. Encourage pupils to consider how these familiar sequences could appear in 'slight disguise' and how they could develop strategies to recognise them.

For examples see the *Framework for secondary mathematics* pages 155 and 159	Use knowledge of related geometrical patterns Use differences to test for types of sequence

The Framework for teaching mathematics: Years 7, 8 and 9, supplement of examples, pages 144 to 159, provides contexts in which pupils could develop mental processes in sequences.

Activities

Building a sequence, using tiles or linking cubes, gives pupils opportunities to think about the construction of each term.

Colour-coding can help to illustrate the structure of the start point and show increments from term to term. Use of colour can also help pupils find the words to describe what they see.

Encourage them to describe the stages of their thinking by suggesting sentence stems, for example:

- I started by thinking of this shape as …
- To get the next shape you …
- The hundredth shape will have …
- The *n*th shape will have ….

Model this process, thinking aloud as you work. Write the rules in words first and then move to symbols. Ask a pupil to give a commentary as you build a sequence of shapes. Illustrate the fact that apparently different algebraic forms can be used to describe the same sequence, depending on how you see each term built up. This equivalence may not be obvious to pupils, so it needs to be discussed explicitly.

This gives pupils opportunities to appreciate that the algebraic forms are equivalent by relating the algebra to the structure of the shape rather than manipulating expressions.

Classifying cards helps pupils to build a mental picture of the different types of sequence. For example, they might choose to sort the cards according to whether the sequences are ascending or descending, and by equal or unequal steps.

A set of cards could include:

- sequences of geometrical patterns
- names of sequences (square numbers, triangular numbers, odd numbers, even numbers, multiples)
- functions relating the term to its position in the sequence (expressed in different ways)
- a list of numbers in a sequence
- a table showing position and terms in a sequence.

For an illustration of the types of card, see the *Framework for secondary mathematics* supplement of examples, pages 146 and 147.

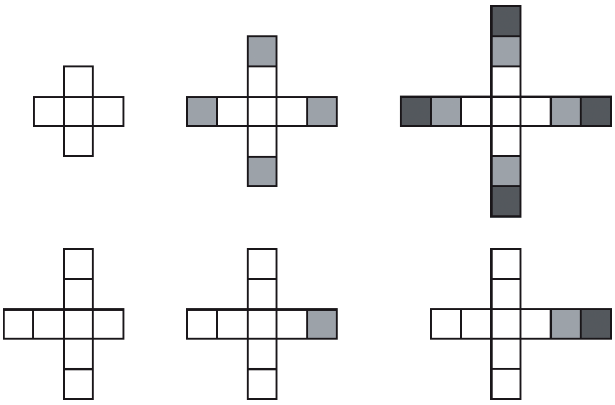
For **Find the function**, pupils use ICT to compose and work with formulae, linking a term to its position in the sequence, for example, T(n) = 5 + 4n.

This can be done on a spreadsheet or a graphing calculator. Pupils work in groups of two pairs. Each pair uses ICT to compose and test a formula. They then give the other pair a short chain of terms in the sequence (these may not be the first terms of the sequence), challenging them to find the function. The guessing pair use ICT to test their attempts at finding the function and finally check their result with the opposite pair.

Discuss the strategies that pairs could use to find a function from a short sequence of terms. Consider exploring differences between consecutive terms, noting the links between these differences and the nature of the function. For an illustration of strategies and ICT applications, see the *Framework for secondary mathematics* supplement of examples, pages 149 to 151.

Snapshots of a sequence can be used to allow pupils to speculate about preceding and subsequent terms. For example, use blocks to show one shape and tell pupils that this is the third in the sequence. Ask what the second and fourth shapes could look like. Share solutions and choose one or two to consider together (see the example below, showing the second, third and fourth terms for two possible sequences).

Generate the rule, in words, for the step from one term to the next. Write the function in symbols for the general term for both sequences. Illustrate both sequences in diagrams and numbers and show that the function gives the same number of blocks for the third term but generates a different sequence.



Term 2 Term 3 Term 4

Functions and graphs

Unless attention is focused on mental processes involved in work on functions and graphs, there is a real risk that pupils will be expected to move rather too quickly from plotting coordinates to tackling challenging generalisations that link algebraic and graphical forms. Pupils require a higher level of thinking to make connections between real-life contexts and the features of a graph. Development of such skills is best supported by collaborative endeavour, allowing pupils the opportunity to share their emerging understanding and to learn from one another. Pupils are better able to tackle challenging problems independently if they have first experienced some success in those areas through interactive group work.

Activities relating to functions and graphs can take two main forms: interpreting graphs or generating graphs. These should be developed alongside each other.

Interpret graphs of functions

Interpreting pre-drawn graphs provides pupils with opportunities to recognise and generalise the relationship between elements in the function and features of the graph. ICT applications are an ideal medium, both for teacher and pupils, because they provide the means for quickly and accurately testing hypotheses about these links.

$y = mx$ $y = c + x$, $y = x + c$ $y = mx + c$	Recognise these graphs for integer values of m and c Note the relationship between families of graphs as values of m and/or c increase or decrease Note which functions represent proportional relationships
$y = ax^2$ $y = x^2 + c$ $y = ax^2 + c$ $y = (x + b)^2$ $y = (x + b)^2 + c$ $y = (x + b)(x + a)$	Recognise these graphs for integer values of a, b and c Note the relationship between families of graphs as values of a and/or b and/or c increase or decrease

Generate graphs of functions

An important skill is the ability to summarise the key features of a graph through a sketch.

This can be developed alongside skills involving graphical calculators or graph-plotting software. In all cases it is crucial to explore problems, discuss results and explain the relationship between the features of a function and the consequent features of the graph.

$y = mx$ $y = c + x$, $y = x + c$ $y = mx + c$	Sketch these graphs for integer values of m and c Explain the relationship between families of graphs as values of m and/or c increase or decrease Explain which functions represent proportional relationships
$y = ax^2$ $y = x^2 + c$ $y = ax^2 + c$ $y = (x + b)^2$ $y = (x + b)^2 + c$ $y = (x + b)(x + a)$	Sketch these graphs for integer values of a, b and c Explain the relationship between families of graphs as values of a and/or b and/or c increase or decrease

Interpret graphs arising from real-life problems

Consider using graphs from other subject areas, such as science or geography, or those that appear in newspapers, other published material or on the internet. Ask pupils to explain what they think the graph might be about. Discussion about the shape of a graph and how it is related to the variables and the context represented supports pupils' understanding.

linear conversion	a single straight line, interpreting the meaning of points and sections
distance–time	linear sections, interpreting the meaning of points and sections
temperature change	curved sections, interpreting the meaning of points and sections

Generate graphs arising from real-life problems

Use ICT to generate graphs of real data, including application data from other subject areas. Focus on the degree to which the graph is an accurate interpretation of a real situation (recording temperature change) or part of a mathematical model (distance–time for a cycle journey). Hypothesising about graphs without scales and headings can draw attention to the way in which different scales and starting points can lead to different interpretations.

linear conversion	a single straight line, interpreting the meaning of points and sections
distance–time	linear sections, interpreting the meaning of points and sections
temperature change	curved sections, interpreting the meaning of points and sections

The *Framework for secondary mathematics* supplement of examples, pages 164 to 177, provides contexts in which pupils could develop mental processes in sequences.

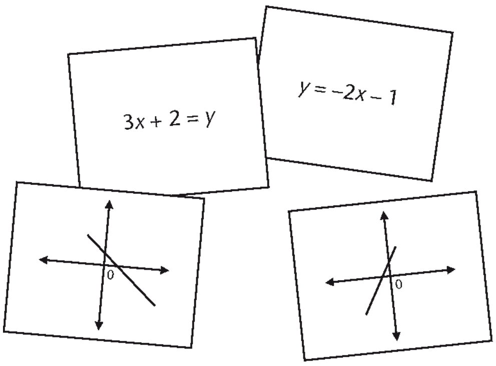
Activities

Matching activities can develop and consolidate pupils' understanding. Provide cards showing tables of values, sets of points, equations and graphs of lines or curves. Model this task, thinking aloud to provide a running commentary. Invite a few pupils to do the same. Agree on helpful prompts or questions such as:

- Do both values increase?
- Which point is important to note?
- What would be a good checking value?

Resources to support an example of a *Functions and graphs* matching activity are given on pages 25–31. Encourage pupils, in pairs, to challenge each other's choices and decisions. Circulate but do not intervene unless pupils are really stuck. Take notes of interesting tactics and discussion points to feed into the plenary. Put pairs together into fours where you can see that a discussion about different choices would be useful.

Wise words is a versatile task suitable for developing understanding of most visual forms. Pupils work in pairs with a set of up to eight cards. For example, four cards could be:



Pairs compose one or two statements to describe a card, which another pair must try to identify. Each statement may use only one key word and may not say what form the representation takes (for example, whether it is a table, a set of points, a function or a graph). Key words might be 'gradient' and 'intercept' and a statement describing the graph on the bottom left card (above) could be: 'The gradient is negative.'

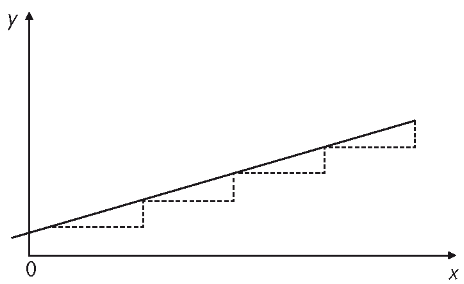
Each pair passes their statements to another pair, for them to work out which card is being described. The second pair can write one 'yes/no' question and, after getting a response to this, they must identify the card. A suitable question could be: 'Is the intercept positive?'

The design of the cards, the number of cards and the key words makes this a rich and adaptable activity, which engages pupils in discussion and forces them to consider the precision of the language they are using.

Say what you see helps pupils to understand the process of drawing graphs.

Use ICT to plot the graph of a linear function, building up the line step by step, giving a running commentary. Use dotted lines to show changes in the *x*-values (horizontal) and corresponding changes in the *y*-values (vertical). As you plot points and trace the graph, ask pupils to say what they see. For example:

- The steps all look the same
- The line is made up of smaller lines all the same
- It could go on for ever … both ways.



Ask if other lines could grow in the same sort of steps. Draw some more parallel lines in the same way. Give groups copies of a worksheet, prepared by pasting an A4 sheet (showing a family of parallel straight lines on unmarked axes) in the centre of a blank A3 sheet. Ask pupils to use graphing calculators to explore a possible interpretation of these lines and to annotate this image.

Note: This task is open to many levels of response. The absence of labels and scales allows groups to generate different functions, making for a rich plenary. Some pupils or groups may use coordinates or tables to justify the functions illustrated. Others may take the opportunity to generalise and offer more than one solution.

Living graphs is an activity in which prepared images prompt pupils to think around a 'real-life' context. It leads to pupils composing their own interpretations and ultimately constructing the graph for themselves. An example, *The speed of a racing car,* is given on page 30. Choose a suitable subset of the statement cards. Use the statement cards to model the process, showing that they can be placed in various positions on the graph, and to give pupils confidence. Ensure that the justification for positioning each statement is clear. Pupils have to decide where the statement would be best positioned from the information they have been given and justify their reasoning to the group.

Some pupils may benefit from giving a 'running commentary' and using this to help them to position the cards. As an extension, one pair could devise a track and commentary and challenge another pair to draw the speed graph for it.

Each of these activities engages pupils with the interpretation of prepared graphical forms. The next stage is for pupils to generate the graph for themselves from another form of representation. In this way pupils build up a mental picture of the links between a table or function and the graph, or between aspects of the 'real-life' context and sections of the graph. Each of the activities can build towards this stage. For example, pupils may create:

- an additional graph that belongs with the set of cards
- another line that belongs to the existing family of lines
- a replacement section of the real-life graph if one small change is made to the context.

Adapted from 'Sketching graphs from pictures', from *The Language of Functions and Graphs* by Malcolm Swan and the Shell Centre Team, 1985.

Examples

Family of equations

commutative

$3 + 5 = 8$ $\quad\quad$ $5 + 3 = 8$

inverse $\quad\quad$ inverse

$3 = 8 - 5$ $\quad\quad$ $5 = 8 - 3$

commutative

$5 + x = 12$ $\quad\quad$ $x + 5 = 12$

inverse $\quad\quad$ inverse

$5 = 12 - x$ $\quad\quad$ $x = 12 - 5$

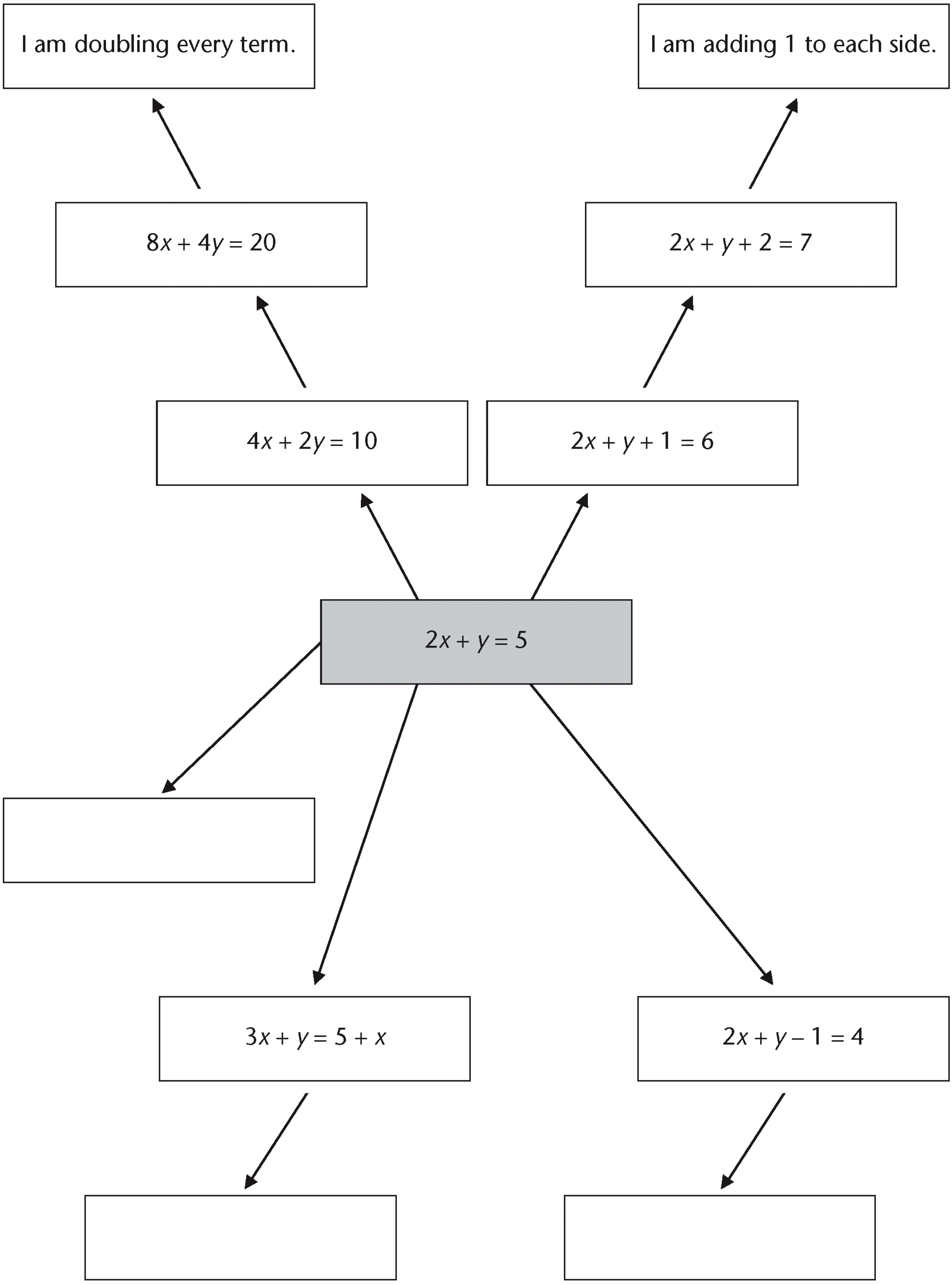
commutative

$(x + 4) \times 3 = 27$ $\quad\quad$ $3 \times (x + 4) = 27$

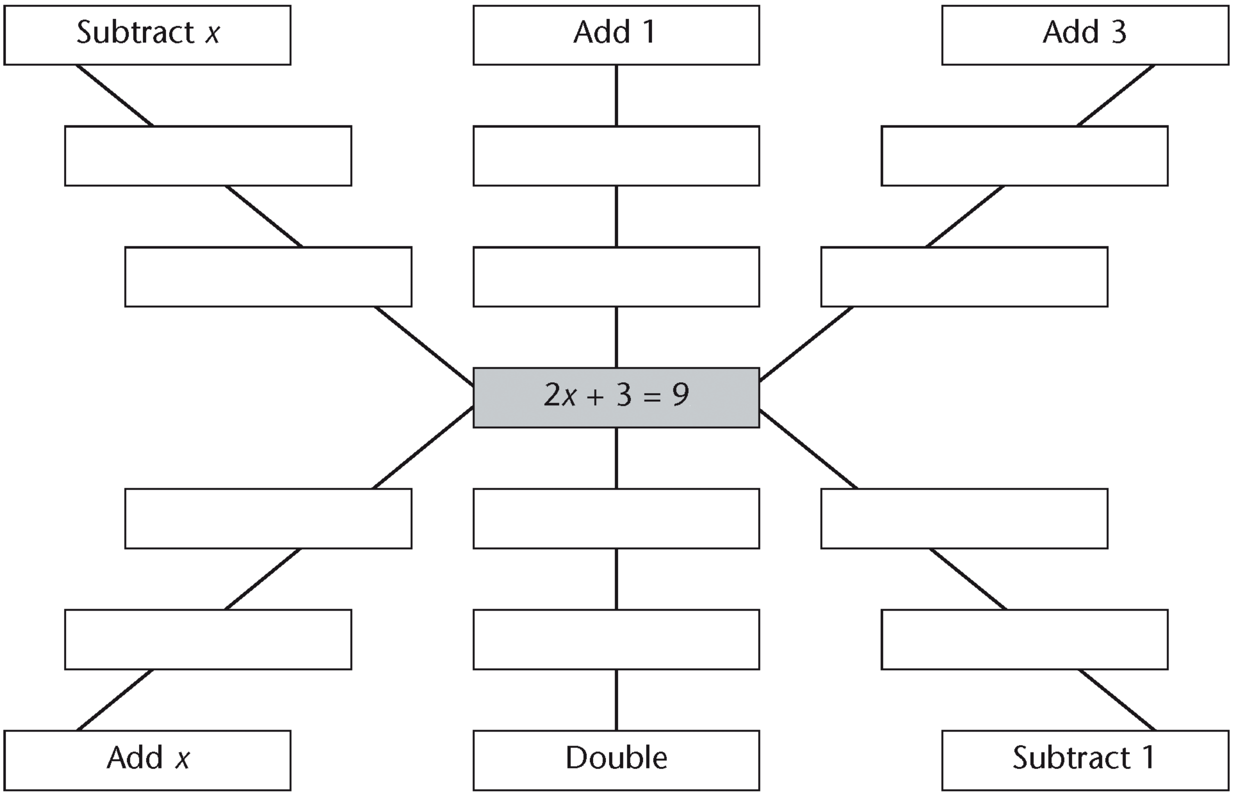
inverse $\quad\quad$ inverse

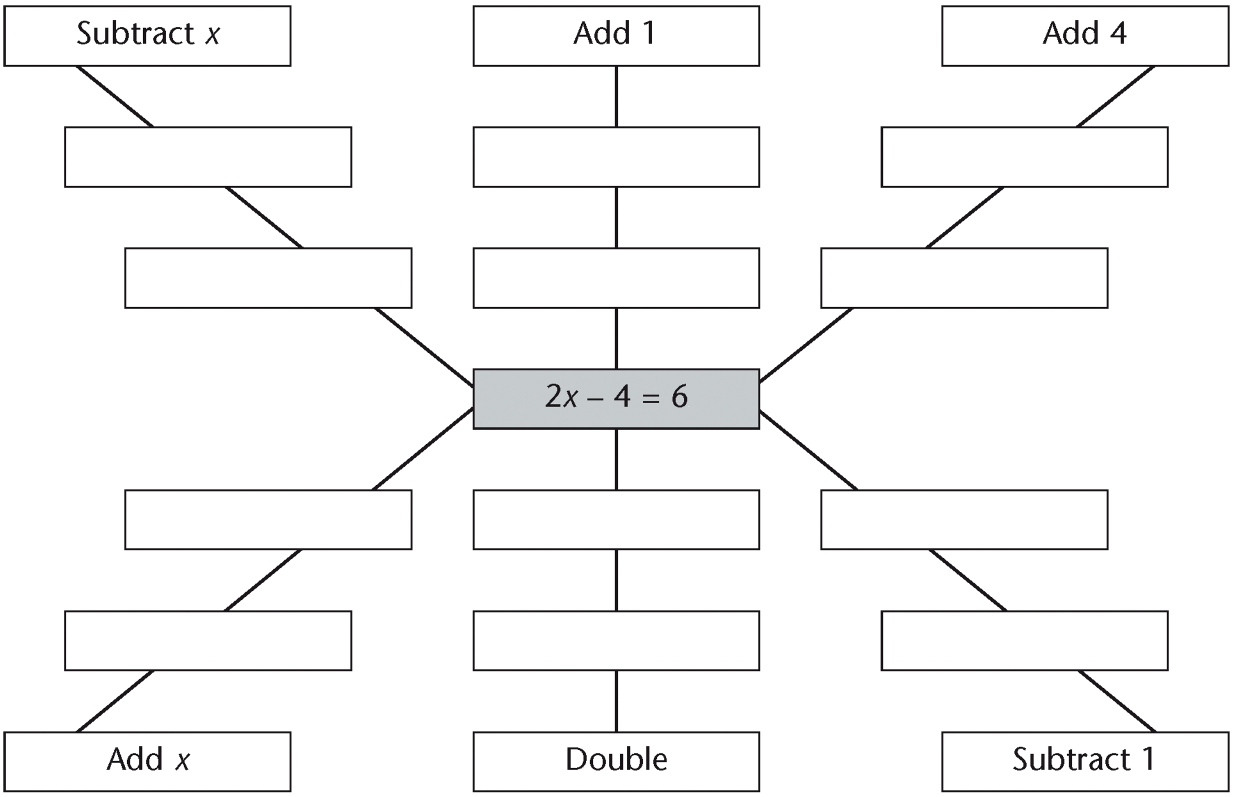
$(x + 4) = 27 \div 3$ $\quad\quad$ $3 = 27 \div (x + 4)$

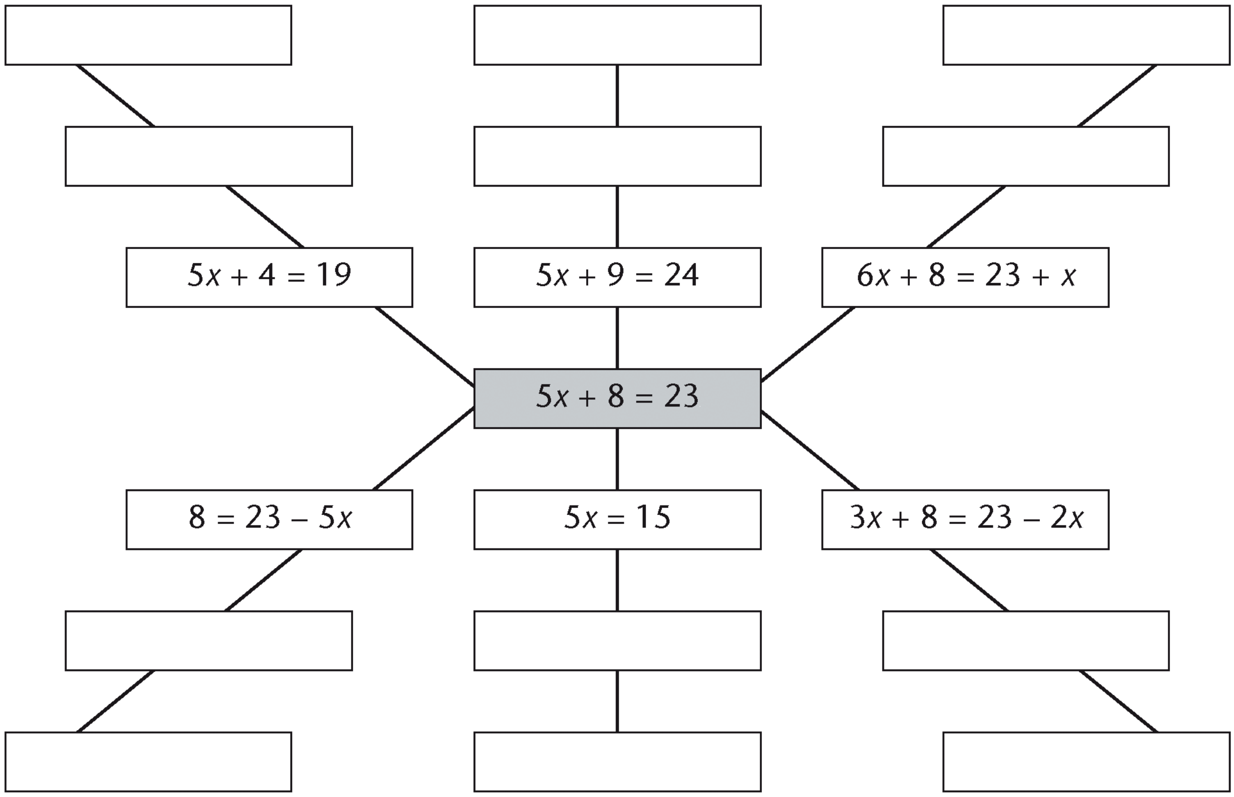
Clouding the picture: algebra 1



Clouding the picture: algebra 2






$5x + 4 = 19$
$5x + 9 = 24$
$6x + 8 = 23 + x$
$5x + 8 = 23$
$8 = 23 - 5x$
$5x = 15$
$3x + 8 = 23 - 2x$

Matching: functions and graphs

Function cards

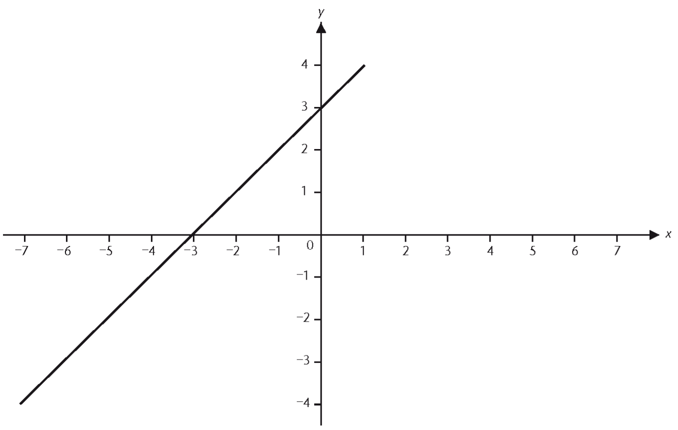
$y = x + 3$	$x + y = 3$	$y = 2x - 3$
$y = 3x$	$y = \frac{1}{2}x + 3$	The line parallel to $y = 2x + 1$ and passing through the point (0.3)
$y = x^2 + 3$	$x = 3$	
$y = 3$	$3x + 4y = 12$	

Group answer sheet

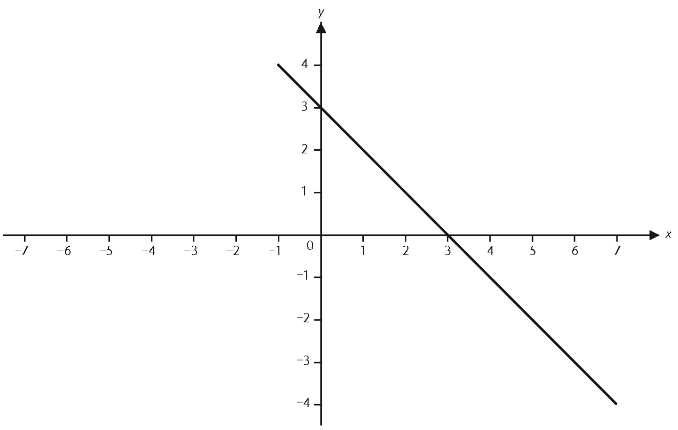
Graph of function	Equation	What strategy did you use to match them? What were the clues?	Key words and expressions
A			x-axis y-axis origin gradient(m) intercept coordinates $y = mx + c$ equation substitute solution linear
B			
C			
D			
E			
F			
G			
H			
I			

Graph cards A–C

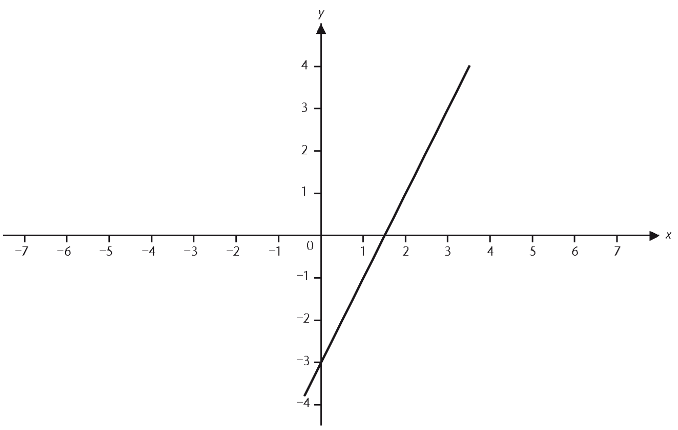
Graph A



Graph B

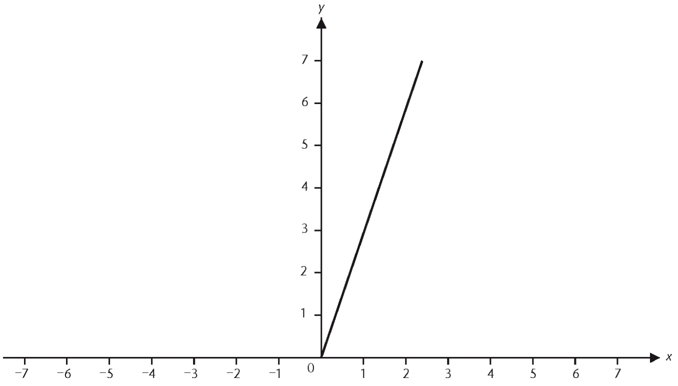


Graph C

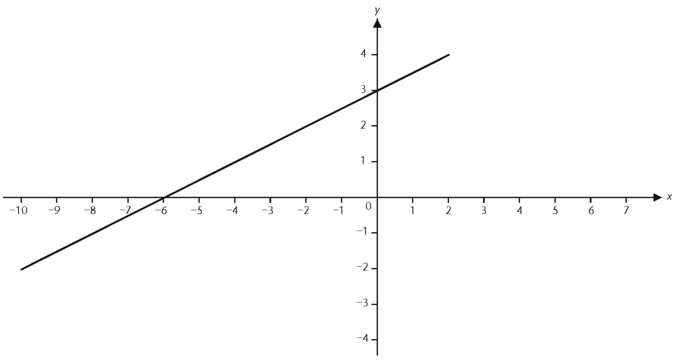


Graph cards D–F

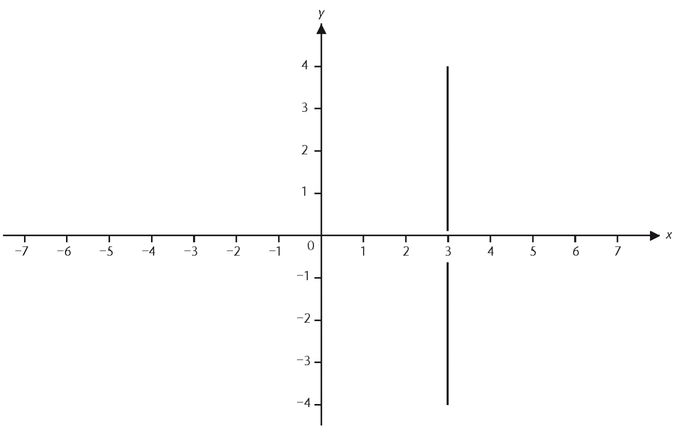
Graph D



Graph E

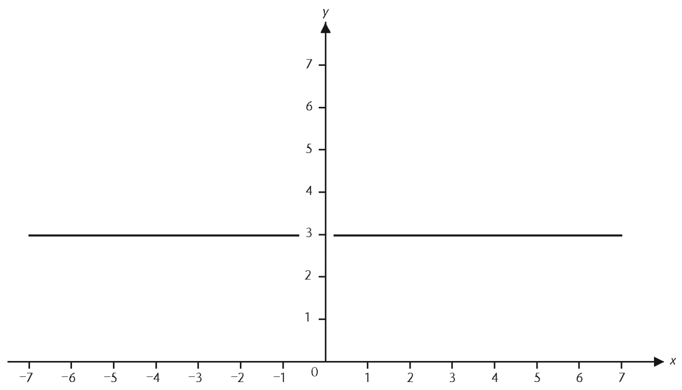


Graph F

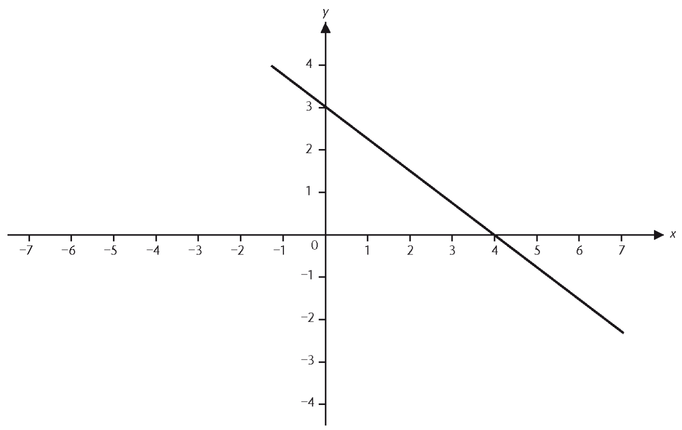


Graph cards G–I

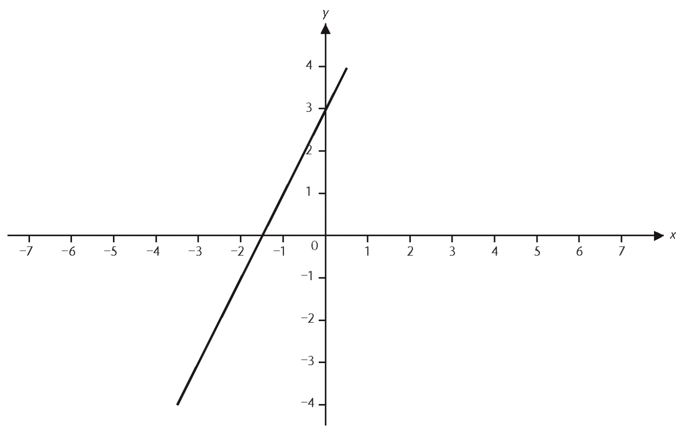
Graph G



Graph H

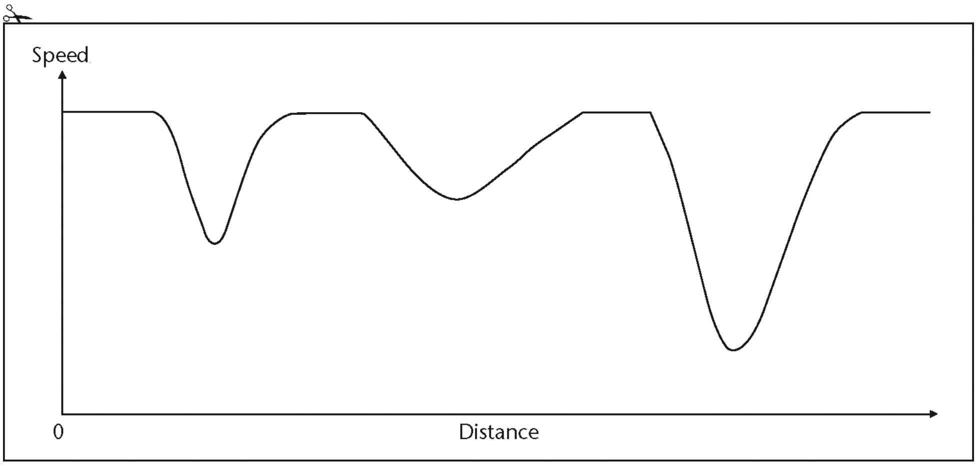


Graph I



Living graphs: the speed of a racing car (lap 2)

Enlarge this graph and make a copy for each group.



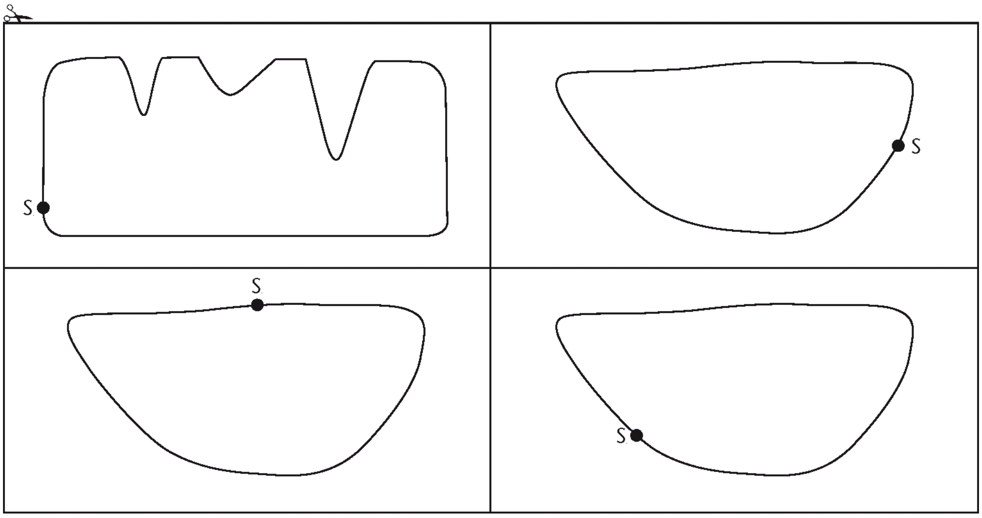
Statement cards

Copy these and cut them into cards, to be tacked in position on the large graph.

The car is travelling at a constant speed.	The car speeds up very rapidly.
The car has stopped.	The car is slowing down.
The car slows down very rapidly.	The car is speeding up.
The car is travelling at the greatest speed shown on the graph.	The car is just arriving on a straight bit of track.
The lowest speed shown on the graph.	The car is meeting a bend.
The car is going round a tight bend.	The car is meeting the worst bend.

Possible circuits

The starting point is marked S, the car travels clockwise. Copy these and cut them into cards, to be matched to the graph and statements.



Adapted from 'Sketching graphs from pictures', from *The Language of Functions and Graphs* by Malcolm Swan and the Shell Centre Team, 1985.

Acknowledgements

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Activity on pages 20 and 30–31 adapted from 'Sketching graphs from pictures', from *The Language of Functions and Graphs* by Malcolm Swan and the Shell Centre Team, 1985. © Shell Centre Publications. **www.mathshell.com**. Used with kind permission.

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