A Level Mathematics Working Group

Report on Mathematical Problem Solving, Modelling and the Use of Large Data Sets in Statistics in AS/A Level Mathematics and Further Mathematics

December 2015

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Contents

1 Introduction

New AS/A level qualifications in mathematics and further mathematics will be taught in schools and colleges from September 2017. The content for these qualifications was developed by the Department for Education (DfE), in collaboration with the A Level Content Advisory Board (ALCAB), and was published in December 2014.

To support the development of new qualifications and assessments reflecting this new content, we convened the A level mathematics working group in March 2015 to provide expert advice in the areas of mathematical problem solving, modelling and the use of large data sets in statistics.[1]

This report is the output of that group. It contains advice from group members on how these key aspects of the new content could be assessed and provides examples of questions. It is a rich source of information for exam boards to consider when designing their assessments, and it will be used as one of the sources of evidence for us to consider in making regulatory decisions. Alongside other sources of evidence, such as consultation responses, this report will be used to:

- support finalisation of assessment objectives and weightings for AS/A level mathematics and further mathematics;
- inform the development of regulatory documentation, including Conditions, Requirements and statutory guidance, such as the technical interpretations of the assessment objectives;
- support the development of high-quality assessments, particularly in relation to mathematical problem solving.

This report includes views of members of the A level mathematics working group and does not necessarily represent the views of any one organisation or Ofqual. Neither the working group nor its output has regulatory status.

The examples in this document are provided for illustrative purposes only, and neither questions nor mark schemes have gone through the full review process for use in live examinations. The examples have been drawn from various sources and thus reflect different approaches to assessment.

[1] The terms of reference for the working group, including details of membership, can be found in appendix 1.

2 Mathematical problem solving

2.1 Detailed content expectations and definitions

Mathematical problem solving is a key feature of GCSE and AS/A level mathematics. It is clear from the subject-content documents for these qualifications that mathematical problem solving is not just for the highest-achieving candidates: it is a core part of mathematics that can and should be accessible to the full range of candidates. For AS and A level mathematics and further mathematics, mathematical problem solving is described in overarching theme 2 (paragraph 7 of the mathematics content document, and paragraph 10 of the further mathematics content document). These overarching themes are a set of descriptions intended to inform and shape the teaching and learning of AS and A level mathematics and further mathematics.

While the importance of mathematical problem solving is clear, defining what it means in the context of examinations is complex. There is a range of views on what it ought to include and how it might be captured in assessments.

One way to explore how best to assess problem solving is to consider the possible attributes of assessment problem solving tasks.[2] The following list contains examples of some of these attributes. These would be expected to be present in tasks that focus primarily on the assessment of problem solving, but may also arise in questions designed primarily to assess other aspects of the detailed subject content and that contain a problem solving element. It is not necessary for every problem solving task to exhibit all of the following attributes, although at least one attribute should apply for a task to be regarded as problem solving:

A. Tasks have little or no scaffolding: there is little guidance given to the candidate beyond a start point and a finish point. Questions do not explicitly state the mathematical process(es) required for the solution.
B. Tasks provide for multiple representations, such as the use of a sketch or a diagram as well as calculations.
C. The information is not given in mathematical form or in mathematical language; or there is a need for the results to be interpreted or methods evaluated, for example, in a real-world context.
D. Tasks have a variety of techniques that could be used.

[2] In this document, we refer to problem solving tasks, which, in this context, we have understood to mean a set of requirements focusing on one problem. These tasks may be broken down into a number of steps or parts (that is, items), but this should not undermine the expectation for AS/A level candidates to demonstrate their ability to solve problems as a coherent process.

E. The solution requires understanding of the processes involved rather than just application of the techniques.
F. The task requires two or more mathematical processes or may require different parts of mathematics to be brought together to reach a solution.[3]

Sometimes problem solving is described as being assessed in tasks that are unfamiliar or non-routine. For problem solving to be validly assessed, tasks must not become formulaic or predictable over time, nor be reduced to a learnt routine. Given the range of candidates to whom AS/A level assessments must cater, and the difficulty associated with knowing with certainty what may or may not be familiar to such a large cohort, there is a challenge for assessment developers to continue to develop, over time, new and unfamiliar tasks.

Whilst it is expected that AS and A level examinations will include tasks that allow students to be assessed on their ability to solve mathematical problems as a coherent process, aspects of problem solving can also be validly assessed in isolation alongside other knowledge and skills. This would, in principle, support the effective assessment of problem solving across the full range of candidate performance and the examinations as a whole.

The idea of problem solving tasks that address the entirety, or large parts of, the problem solving cycle is described in the overarching themes (OT) in the subject content for AS/A level mathematics and further mathematics.

> OT2 Mathematical problem solving
>
> OT2.6 Understand the concept of a mathematical problem solving cycle, including specifying the problem, collecting information, processing and representing information and interpreting results, which may identify the need to repeat the cycle.[4]

Similarly, we could consider how candidates could approach a problem and aim to design tasks that require them to work through this process. This could be described as a “choose – organise – sustain” model that requires candidates to choose an

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[3] Not all of these attributes would be required within a single task to establish it as problem solving. Neither does the presence of one or more attributes within a task automatically imply problem solving is taking place.
[4] Subject content requirements: www.gov.uk/government/uploads/system/uploads/attachment_data/file/472581/GCE_AS_and_A_level_subject_content_for_mathematics.pdf and www.gov.uk/government/uploads/system/uploads/attachment_data/file/472580/GCE_AS_and_A_level_subject_content_for_further_mathematics.pdf

approach, organise the available information and sustain the process of using the information and their own knowledge to solve the problem. The need to choose an approach reflects the fact that the method required is neither given nor made immediately clear by the task itself. The need to use the information provided and the candidate's own knowledge reflects the fact that different aspects of mathematics may need to be brought together to reach a solution. On some occasions, it might also be appropriate for the process to end with evaluation, where candidates consider the effectiveness of their approach in solving the problem.

As described above, there may also be instances where it is appropriate to assess small parts of problem solving within questions or items targeted primarily at other assessment objectives, but all candidates must have the opportunity to attempt complete problems presented in an unstructured manner requiring the use of multiple parts of the problem solving cycle.

2.2 Problem solving exemplars[5]

These problem solving tasks were selected by the working group as exemplars. For each task a supporting commentary has been provided. This identifies the features of the task that make it problem solving. Draft mark schemes are provided, and further comments on the implications for mark schemes are addressed in the following section of this report.

[5] These examples should not be regarded as exam-ready (see Introduction). Their key purpose is the illustration of the problem solving attributes. In addition, although the examples provided are drawn from the area of pure mathematics, this does not mean that problem solving tasks cannot be set in context.

Problem solving exemplar 1

A circle, centre C (1, 1), touches both axes, as shown in Fig. 1. AB is a tangent to the circle. The triangle OAB is isosceles.

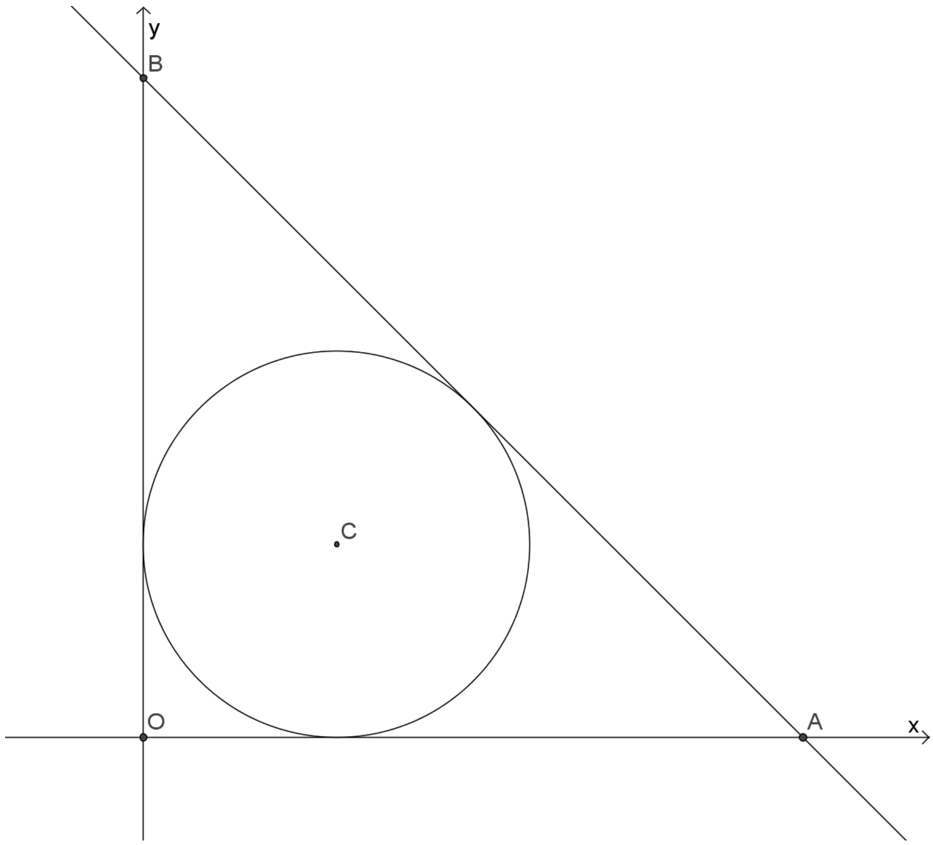


Fig. 1

Find the equation of AB, giving your answer in exact form. **[8]**

Problem solving exemplar 1 – mark scheme

<table>
<tr>
<td>1</td>
<td></td>
<td>Radius = 1 soi
Use of right-angled isos triangle with OC, radius and part of axis
OC = $\sqrt{2}$
Distance O to point of contact = $1+\sqrt{2}$</td>
<td>M1
M1

A1
A1</td>
<td></td>
<td rowspan="3"><u>Method 2</u>
M1 for radius = 1 soi
M1 for equation of circle is $(x-1)^2+(y-1)^2=1$
M1 pt of contact is (*a*, *a*) oe
M1 $(a-1)^2+(a-1)^2=1$
M1 $a=1\pm\sqrt{\frac{1}{2}}$
A1 *a* > 1 so $a=1+\sqrt{\frac{1}{2}}$
M1 AB has gradient –1
A1 Equation $x+y=2+\sqrt{2}$

<u>Method 3</u>
M1 two tangents to circle from a point equal
M1 sides of triangle are 1 + *c*, 1+ *c*, 2*c*
M1 $\frac{1+c}{2c}=\frac{1}{\sqrt{2}}$ or correct Pythagoras or other correct trig statement
M1 $2c-c\sqrt{2}=\sqrt{2}$
A1 $c=\frac{\sqrt{2}}{2-\sqrt{2}}$ oe
A1 OA (or OB) = $2+\sqrt{2}$ oe
M1 AB has gradient –1
A1 Equation $x+y=2+\sqrt{2}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>One coordinate of pt of contact is $\frac{1+\sqrt{2}}{\sqrt{2}}$
Pt of contact $\left(\frac{\sqrt{2}+2}{2}, \quad \frac{\sqrt{2}+2}{2}\right)$
oe</td>
<td>M1

A1</td>
<td>

Must be exact</td>
</tr>
<tr>
<td></td>
<td></td>
<td>AB has gradient –1
Equation $x+y=2+\sqrt{2}$</td>
<td>M1
A1
[8]</td>
<td>
Exact and rationalised</td>
</tr>
</table>

Problem solving exemplar 1 – commentary

This question could be included in the assessment of AS mathematics as no knowledge beyond AS is required to solve it. Method 3 includes some mathematics that is not in A level but is in GCSE (this is assumed knowledge for A level). Several of the problem solving attributes are exhibited:

A there is no scaffolding;
B multiple representations are involved;
D the problem can be solved in several ways – three methods are suggested in the mark scheme;
E the question requires more than simple application of techniques;

F several areas of mathematics are used in the question: coordinate geometry, surds and (possibly) trigonometry.

Problem solving exemplar 2

Find the complete set of values of x for which $|x+1| > 2|x-1|$.

Problem solving exemplar 2 – mark scheme

Determining the critical values and identifying the interval can be carried out in either order or alongside each other.

Graphical approach:

α	B1	Sketch (on a single set of axes) of $y = \|x+1\|$ and $y = 2\|x-1\|$ with vertices on the *x*-axis (the first to the left of the second) and the second curve steeper than the first, the curves each being made up of two straight-line segments.
β	B1	Identification of the two points of intersection where $a+1 = -2(a-1)$ and $b+1 = 2(b-1)$. [This might be done by emphasising the points of intersection instead of writing down the equations, but in that case the intersecting branches of the modulus curves should be labelled free of moduli.]
γ	M1	Complete attempts at solving the linear equations for both the critical values.
δ	A1	Correct critical values $a = \frac{1}{3}$ and $b = 3$ [but not necessarily identifying which is which].
ε	M1	Identification of the correct interval for the inequality as $a < x < b$ or $x \in (a, b)$ or $x \in]a, b[$ or by emphasising *x*-axis or by emphasising upper curve or by shading.
ζ	A1	Correct final answer as $\frac{1}{3} < x < 3$ or $x \in \left(\frac{1}{3}, 3\right)$ or $x \in \left]\frac{1}{3}, 3\right[$.

Critical values by squaring (alternative mark scheme for marks β and γ, then award δ as above):

β′	M1	Identifies the equation $(x+1)^2 = \left(2(x-1)\right)^2$ and expands (or uses differences of two squares to factorise). [Can be implied by the next mark.]
γ′	M1	Complete attempt to solve $(x+1)^2 = \left(2(x-1)\right)^2$ obtaining exactly two roots.

Interval determined using sign table (alternative mark scheme for marks α and ε, then award ζ as above):

α′ M1 Use of sign table with at least three intervals with either clear comparison of $|x+1|$ and $2|x-1|$ or clear consideration (either way round) of the difference between $|x+1|$ and $2|x-1|$.

ε′ A1 Correct signs for the three intervals $x < a,\ a < x < b,\ b < x$. [Do not penalise if these intervals are subdivided, as long as the correct signs are obtained.]

Problem solving exemplar 2 – commentary

In this question candidates have to choose an approach to solve it without guidance, select the information necessary to find the critical values and then pursue their approach to a conclusion. Several of the problem solving attributes are exhibited:

A there is no scaffolding;
B a (sketch) graph is needed as well as algebra;
D there is a choice of technique between using the graph and squaring to find the critical values, and between using the graph and a sign table for determining which interval satisfies the inequality;
F the question requires two processes: one to find the critical values and one to find the intervals.

Problem solving exemplar 3

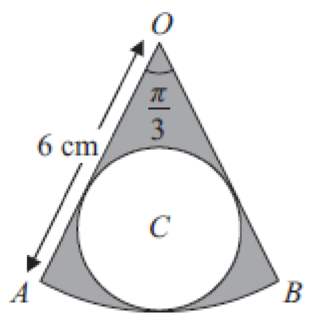


Figure 1

The shape shown in Figure 1 is a pattern for a pendant. It consists of a sector *OAB* of a circle centre *O*, of radius 6 cm, and angle *AOB* = $\frac{\pi}{3}$. The circle *C*, inside the sector, touches the two straight edges, *OA* and *OB*, and the arc *AB* as shown.

Find

(*a*) the area of the sector *OAB*, **(2)**
(*b*) the radius of the circle *C*. **(3)**

The region outside the circle *C* and inside the sector *OAB* is shown shaded in Figure 1.

(*c*) Find the area of the shaded region. **(2)**

Problem solving exemplar 3 – mark scheme

Question No.	Scheme		Marks
(a)	$\frac{1}{2}r^2\theta = \frac{1}{2}(6)^2\left(\frac{\pi}{3}\right) = 6\pi$ or 18.85 or awrt 18.8 (cm)^2	Using $\frac{1}{2}r^2\theta$ (See notes)	M1
		6π or 18.85 or awrt 18.8	A1
			[2]
(b)	$\sin\left(\frac{\pi}{6}\right) = \frac{r}{6-r}$	$\sin\left(\frac{\pi}{6}\right)$ or $\sin 30^\circ = \frac{r}{6-r}$	M1
	$\frac{1}{2} = \frac{r}{6-r}$	Replaces sin by numeric value	dM1
	$6 - r = 2r \Rightarrow r = 2$	$r = 2$	A1 **cso**
			[3]
(c)	Area $= 6\pi - \pi(2)^2 = 2\pi$ or awrt $6.3\ (\text{cm})^2$	their area of sector – πr^2	M1
		2π or awrt 6.3	A1 **cao**
			[2]
			7

Problem solving exemplar 3 – commentary

The problem solving part of this question is part (b).

The question does not provide guidance on the techniques required to solve the problem and also requires candidates to draw together a number of different techniques. Much of the mathematical content used is from GCSE but, put into a problem solving context, the question is appropriate for A level.

Several of the problem solving attributes are exhibited:

A There is little or no scaffolding and the question does not explicitly state the mathematical process(es) required for the solution: there is little guidance given to the candidate beyond a start point and a finish point.

B Multiple representations are involved, such as the use of a sketch or a diagram as well as calculations. Although the diagram is given and therefore not drawn by the candidates, they are unlikely to get the solution without engaging with the diagram.

F The question requires two or more mathematical processes.

Problem solving exemplar 4

Find the equation of a cubic graph that touches the *x*-axis at the point (3, 0) and goes through the point (0, –1) on the *y*-axis. Explain whether it is possible to find more than one such cubic
graph.

[8]

Problem solving exemplar 4 – mark scheme

		Factor $(x – 3)$ oe Repeated factor $(x – 3)$ oe	**M1** **A1**		<u>Special cases if no marks awarded for equation of graph</u> **SC1** for cubic through (0, –1) **SC2** for quadratic which meets both conditions
		$y=(x+a)(x-3)^2$ *x* = 0 substituted $-1=9a$ so $a=-\frac{1}{9}$ $y=\left(x-\frac{1}{9}\right)(x-3)^2$ oe	**M1** **M1** **A1**	Third factor Any cubic graph which meets both conditions	A general cubic, rather than a specific one, scores full marks here
		The cubic must have a repeated factor $(x – 3)$ to touch the *x*-axis. Suppose the third factor is different to *their* $\left(x-\frac{1}{9}\right)$ e.g. any other linear factor (not a multiple of *x*) Stretching the graph parallel to the *y*-axis would make it go through (0, –1)	**E1** **E1** **E1** **[8]**	Could be implied	**OR** **M2** for finding a different equation that meets the conditions **A1** so yes it is possible

Problem solving exemplar 4 – commentary

Several of the problem solving attributes are exhibited:

A there is no scaffolding;

B multiple representations are likely to be needed;

D there are different ways to tackle the problem;

E understanding the mathematical processes involved is necessary;

F different aspects of mathematics are brought together.

Problem solving exemplar 5

A curve is defined by the parametric equations

$$x = 3 + 2t \ , \ y = 2 - \frac{3}{t}$$

Find the value of *t* at the point where the normal to the curve at (9, 1) crosses the curve again. **[9 marks]**

Problem solving exemplar 5 – mark scheme

	Solution	Marks	Total	Comments
	$x = 3 + 2t \qquad y = 2 - \frac{3}{t}$			
	$\frac{\mathrm{d}x}{\mathrm{d}t} = 2 \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{3}{t^2}$	M1		PI both attempted and at least one correct
	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2t^2}$	M1		Chain rule
	At (9, 1), $t = 3$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{6}$	A1		CSO PI by c working with $-\frac{\mathrm{d}x}{\mathrm{d}y}$ and getting correct gradient for normal
	Gradient of normal at *P* is −6	B1F		
	Eqn of normal at (9, 1): $y - 1 = -6(x - 9)$ $y + 6x = 55$	A1		CSO ACF
	Normal cuts curve when $2 - \frac{3}{t} + 6(3 + 2t) = 55$	M1		Subst parametric eqns in eqn of normal
	$\Rightarrow\ 12t^2 - 35t - 3 = 0$	A1		Forming correct quadratic eqn
	$\Rightarrow\ (12t + 1)(t - 3) = 0$	m1		Factorising or using quadratic formula to solve c's quadratic equation

	$t=3$ [original pt (9, 1)] or $t=-\dfrac{1}{12}$ Normal crosses curve again at point where $t=-\dfrac{1}{12}$	A1	9	

	Alternative: Cartesian equation of curve: $y=2-\dfrac{6}{x-3}$	(M1)		Eliminate t; condone numerical and sign slips; accept unsimplified
	$\dfrac{dy}{dx}=\dfrac{1}{6}\dfrac{6}{(x-3)^2}$	(M1)		Valid method to find dy/dx for $y=\pm a\pm\dfrac{b}{cx\pm d}$ OE
	At (9, 1) , $\dfrac{dy}{dx}=\dfrac{1}{6}$	(A1)		CSO
	Gradient of normal at *P* is −6	(B1F)		
	Eqn of normal at (9, 1): $y=55-6x$	(A1)		CSO ACF
	$2-\dfrac{6}{x-3}=55-6x$; $6x^2-71x+153=0$	(M1)		Solving normal and curve to form quadratic
	$(x-9)(6x-17)=0$	(M1)		Factorising or using quadratic formula to solve c's quadratic equation
		(A1)		Obtaining $x=9$ and 17/6 then equating 17/6 to $3+2t$
	$\dfrac{17}{6}=3+2t\Rightarrow t=-\dfrac{1}{12}$	(A1)		
	Total		**9**	

Problem solving exemplar 5 – commentary

This question addresses specification content B3, B7, C3, G3 and G5.

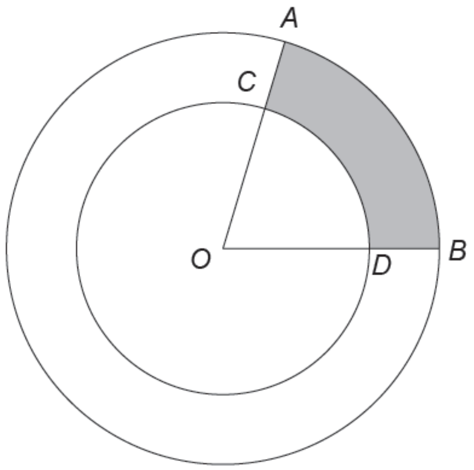
This unstructured question on parametric equations is an example of problem solving. Several of the problem solving attributes are exhibited:

A There is no scaffolding.

D There are different ways to tackle the problem: apply parametric methods to find $\dfrac{dy}{dx}$ as a function of *t*, moving on to find the equation of the normal or eliminate *t* at the outset.

E/F A number of mathematical processes must be used with understanding. There is further choice: to either substitute the parametric equations into the equation of the normal to form and solve a quadratic in *t* or to use the Cartesian equation of the curve and substitute that into the equation of the normal to form and solve a quadratic in x.

Problem solving exemplar 6



The diagram shows two concentric circles with a common centre O. The radius of the larger circle is 7 cm and the radius of the smaller circle is 4 cm. The points A and B lie on the larger circle and OA and OB cut the smaller circle at the points C and D respectively. The area of the shaded region $ACDB$ is 23·1 cm². Find the perimeter of $ACDB$. [6]

Problem solving exemplar 6 – mark scheme

Denoting $A\hat{O}B$ by θ,

Area of sector AOB = $\frac{1}{2} \times 7^2 \times \theta$

Area of sector COD = $\frac{1}{2} \times 4^2 \times \theta$	(at least one correct)	M1
$\frac{1}{2} \times 7^2 \times \theta - \frac{1}{2} \times 4^2 \times \theta = 23{\cdot}1$	(f.t candidate's expressions for the areas of the sectors)	M1
$\theta = 1{\cdot}4$	(c.a.o.)	A1
CD = 5·6 cm, AB = 9·8 cm	(both values, f.t candidate's value for θ)	B1
Use of perimeter of $ACDB = AC + CD + DB + BA$		M1
Perimeter of $ACDB$ = 21·4 cm	(c.a.o.)	A1

Problem solving exemplar 6 – commentary

This question is a pure maths question that is relatively straightforward but does assess aspects of problem solving. Although this aspect of the content is currently assessed in C2 (AS), in the new A level it could only be assessed in the A level, and not in the AS. (See Section E1 in the AS and A level content for mathematics document.)

Candidates would need to introduce the variable for the angle themselves from information provided in the question and in the diagram. The problem is relatively unfamiliar and non-routine.

Several of the problem solving attributes are exhibited:

A It is multi-step and has no scaffolding: there is little guidance given to the candidates. The question does not explicitly state the mathematical processes required for the solution.

E Understanding of the processes involved is required rather than just the application of the techniques.

F Two mathematical processes are required.

2.3 Implications for mark schemes

The exemplars above raise a number of questions about mark schemes. For example, when there are multiple valid approaches to solving the problem, mark schemes must be able to credit all methods appropriately and apply the rewards in a consistent way.

If problem solving involves candidates making decisions about which approach to use to solve a problem and which techniques to employ to do so, mark schemes must be able to reward a range of creditworthy decisions. In traditional mark schemes, marks are generally awarded for method and accuracy but thought must be given to how marks can be awarded for problem solving skills as well. This is particularly the case when credit is to be given for decision making and when evidence of that decision may not always arise in a way that a method or accuracy mark could capture. In addition, when there are different routes towards completing a task, the number and position of decision points may differ and make the reliable application of a traditional method and accuracy mark scheme challenging.

There are other approaches by which mark schemes could effectively capture the key aspects of problem solving. For example, “strategy marks” could be used in addition to method and accuracy marks, to credit when a candidate demonstrates a decision made or an approach chosen, regardless of the extent to which that decision or approach is worked through. Additionally, items where there are multiple valid approaches could be accompanied by multiple mark schemes, one for each approach.[6]

Of course, there would be risks associated with some of these approaches. Mark schemes must support consistent and reliable marking of each question across large numbers of examiners. The greater the professional judgement involved in the marking of a task, the greater the risk to reliable marking. The greater the number of mark schemes available for a single task, the higher the risk that the marking of different approaches might result in a lack of comparability between different approaches to a single task. When a task has multiple solutions, different candidates might demonstrate different skills and knowledge in their responses. Consistent alignment of items and their mark schemes with assessment objectives (including strands and elements) is important in delivering assessments that are appropriately balanced and deliver reliable outcomes.

[6] Another possible approach might be the use of levels-based mark schemes to cover tasks where there are a number of valid approaches. However, it is questionable whether this kind of mark scheme could provide a sufficiently clear and consistent approach to marking in mathematics.

3 Modelling

3.1 Detailed content expectations

Mathematical modelling is covered comprehensively in the subject content in a variety of different contexts. For the purposes of assessment, modelling is currently included in the same assessment objective (AO3) as problem solving. As with problem solving, modelling is encapsulated within the overarching themes in the mathematics and further mathematics content documents, as overarching theme 3.

The content requires candidates to construct their own models, as well as to use known and given models and assumptions, reflecting on the potential impact of their modelling assumptions. While the whole modelling cycle is included in the content, there is no requirement for the cycle to be assessed in its entirety within a single task.

The current practice for assessing modelling remains appropriate, as do the types of questions currently set, but modelling must be considered in all of the contexts described by the content and, where it can be validly and reliably assessed, should extend across more of the modelling cycle.

3.2 Modelling exemplars[7]

Modelling exemplar 1

A demolition worker finds a very deep vertical shaft and wants to estimate the depth of the shaft. To do this he drops a heavy concrete block into the shaft. He hears the block hit the bottom of the shaft 8 seconds after he dropped it.

a) Use this information to estimate the depth of the shaft.

[3 marks]

b) Estimate the speed at which the block hits the bottom of the shaft.

[2 marks]

c) State whether your answer to part (a) is an over- or under-estimate of the actual depth of the shaft. Give two reasons to justify your answer, one of which should make reference to your answer to part (b).

[7] The examples are for illustrative purposes only and have not gone through the full review process for use in live examinations (see Introduction).

[3 marks]

Modelling exemplar 1 – mark scheme

Q	Solution	Marks	Total	Comments
(a)	$s = \frac{1}{2} \times 10 \times 8^2$	M1 A1		Use of constant acceleration equation with initial speed of zero Correct expression
	320	A1	3	Correct distance
(b)	$v = 0 + 10 \times 8$	M1		Correct expression
	$= 80\text{ m s}^{-1}$	A1	2	Correct speed
(c)	Over-estimate	B1		Correct statement
	e.g. Air resistance would decrease the acceleration significantly, as the block is moving very fast.	B1		First appropriate statement
	Sound would take time to travel back up the shaft.	B1	3	Second appropriate statement
	Total		**8**	

Modelling exemplar 1 – commentary

Specification content Q1, Q3 and R3 (but could use Q2).

This question expects the use of constant acceleration equations and an understanding of mathematical modelling. It is based on a real situation that can be found on YouTube, where there is a video of a concrete block being dropped into a very deep hole.

(a) Expects the use of a constant acceleration equation, but also requires candidates to make and use the following modelling assumptions:

- Motion is vertical.
- The block is a particle.
- There is no air resistance.
- There is no water in the shaft.

(b) Expects a simple application of the constant acceleration equations, but is intended to focus the candidates' attention on the fact that the block will be moving very fast and that air resistance would in fact be significant.

(c) Expects the candidates to think about the impact of their assumptions and other factors to evaluate the validity of their estimate and to articulate the reasons for their answers.

This question also displays problem solving attributes:

A The question is not scaffolded.

C The information comes from a real-world situation and the mathematical results must be interpreted in this situation.

D The mathematical processes required in order to obtain a solution are not given to the candidate and there is a choice of techniques that could be used; although constant acceleration questions are expected candidates could approach this solution graphically.

Modelling exemplar 2[8]

A company claims that it receives emails at a mean rate of 2 every 5 minutes.

(a) Give two reasons why a Poisson distribution could be a suitable model for the number of emails received.

(b) Using a 5% level of significance, find the critical region for a two-tailed test of the hypothesis that the mean number of emails received in a 10-minute period is 4. The probability of rejection in each tail should be as close as possible to 0.025.

(c) Find the actual level of significance of this test.

To test this claim, the number of emails received in a random 10-minute period was recorded.

During this period 8 emails were received.

(d) Comment on the company's claim in the light of this value. Justify your answer.

During a randomly selected 15 minutes of play in the Wimbledon Men's Tennis Tournament final, 2 emails were received by the company.

(e) Test, at the 10 per cent level of significance, whether or not the mean rate of emails received by the company during the Wimbledon Men's Tennis Tournament final is lower than the mean rate received at other times. State your hypotheses clearly.

[8] Whilst this content does not fall within the content requirements of A level mathematics, it could be included in A level further mathematics.

Modelling exemplar 2 – mark scheme

Question Number	Scheme	Marks
(a)	Any two of: Emails are independent/occur at random Emails occur singly Emails occur at a constant rate	B1B1d (2)
(b)	*X* ~ Po(4) $\mathrm{P}(X=0)=0.0183$ $\mathrm{P}(X\geq 9)=0.0214$ CR $X=0$; $X\geq 9$	B1B1 (2)
(c)	0.0183 + 0.0214 = 0.0397 or 3.97%	M1A1 (2)
(d)	8 is not in the critical region **or** $\mathrm{P}(X\geq 8)$ = 0.0511 therefore there is evidence that the company's **claim** is true	M1 A1ft (2)
(e)	$H_0 : \lambda = 6$ (or $\lambda = 2$) $H_1 : \lambda < 6$ (or $\lambda = 2$) allow λ or μ Po(6) $\mathrm{P}(X\leq 2)=0.0620$ CR $X \leq 2$ 0.0620 < 0.10 Reject H_0 or Significant. There is evidence at the 10% level of significance that the mean **rate/number/amount** of **emails** received **is lower/has decreased/is less**. Or **fewer emails** are received	B1 M1 A1 M1 dep. A1 cso (5) [13]

Modelling exemplar 2 – commentary

Part (a) requires candidates to give reasons why a Poisson distribution could be a suitable model. The conditions are standard but must be given in the context of the problem rather than as general statements. The remaining parts of the question require the candidate to work with this model.

Modelling exemplar 3

The number, *n*, of lunches served by the canteen of a college with 1000 students depends on the number, *s*, of students attending college that day. The number *n* increases with *s*, but eventually as queues get longer people become less willing to wait and some decide not to join the queue. Fig. 3 shows the graphs of five possible relationships between *n* and *s*.

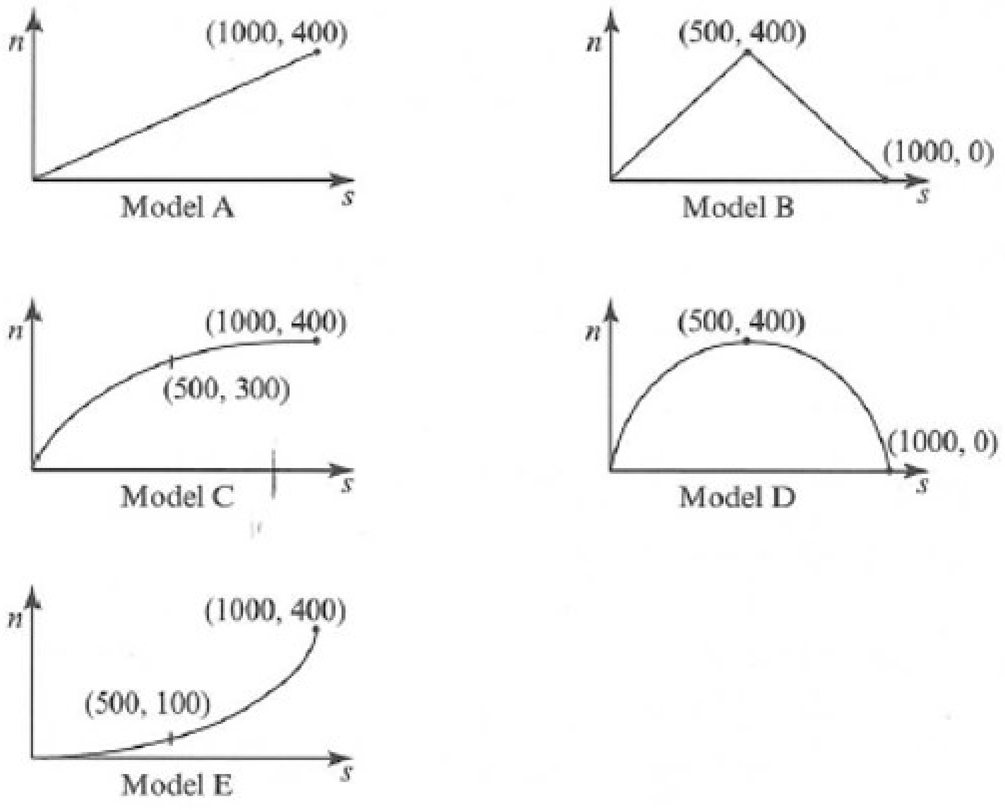


Fig. 3

(a) Explain how Models B and D are inconsistent with the information above.

(b) Explain why Model C is more realistic than Models A and E.

(c) Show that an equation of the quadratic curve used in Model C is

$$n = -\frac{1}{2500}s^2 + \frac{4}{5}s.$$

(d) Use the equation in part (c) to estimate the number of lunches sold when 900 students were in college.

(e) The canteen breaks even if at least 364 lunches are sold. Determine, on the basis of Model C, the range of student attendances for which the canteen should open if it is to break even.

Modelling exemplar 3 – mark scheme

(a)	After s = 500 the models indicate that *n* declines which is not the case.	E1	
(b)	C takes into account the discouraging effect of a long queue but A and E do not.	E1	
(c)	$n = as^2 + bs$ $300 = 250000a + 500b$ $400 = 1000000a + 1000b$	M1	
	$l = -\frac{1}{2500}s^2 + \frac{4}{5}s$	A2	any legitimate method
(d)	396	B1	must be an integer
(e)	$-\frac{1}{2500}s^2 + \frac{4}{5}s - 364 \geq 0$	M1	if written as equation then for M1 somewhere else there must be recognition of the need for an inequality
	$700 \leq s \leq 1300$ *so* $700 \leq s \leq 1000$	A2	1 for 700 as a critical value 1 for 1000 as upper bound

Modelling exemplar 3 – commentary

This question shows how alternative models might be considered. The candidates were constrained eventually to use Model C, but it would have been feasible to leave the final choice of model open and provide alternative mark schemes for those using less appropriate models.

Although the models offered are simple, the question still requires candidates to interpret textual information about the situation mathematically, including relating rates of change to the gradients of graphs, then the graphs to algebraic equations, and finally the algebraic representation back to the context. In so doing, the question exhibits problem solving attribute B as it is likely to involve multiple representations.

Modelling exemplar 4

At time t seconds, the radius of a spherical balloon is r cm. The balloon is being inflated so that the rate of increase of its radius is inversely proportional to the square root of its radius. When $t = 5$, $r = 9$ and, at that instant, the radius is increasing at 1.08 cm s^{-1}.

(i) Write down a differential equation to model this situation, and solve it to express r in terms of t.

(ii) How much air is in the balloon initially?

[The volume of a sphere is $V = \frac{4}{3}\pi r^3$]

Modelling exemplar 4 – mark scheme

Answer	Marks	Guidance
(i) $\frac{dr}{dt} = \frac{k}{\sqrt{r}}$ oe	B2	B1 for each side SR: B1 for $\frac{dr}{dt} \propto \frac{1}{\sqrt{r}}$
Subst conditions to find k	M1	
Sep variable of their diff eqn (or invert) & integrate each side, increasing powers by 1 (or $\frac{1}{r} \to \ln r$)	*M1	Condone absence of '+c'
Subst conditions to find 'c'	DM1	Must involve '+ c' here
Correct value of c (probably 1.8 or –1.8)	A1	
$r = (4.86t + 2.7)^{\frac{2}{3}}$ ISW	A1 [7]	Answer required in form $r = \mathrm{f}(t)$
(ii) Subst $t = 0$ into any version of (i) result to find finite r	M1	
Any V in range $30.5 \leq V \leq 30.55$	A1 [2]	Accept 9.72π or $\frac{243}{25}\pi$

Modelling exemplar 4 – commentary

This is a modelling question from a past paper; it covers content statements G6, H1, H2, H7, H8 and potentially H3 depending on method. It is a bit thin on candidate generation and discussion of assumptions, but it does require translation from context, construction of the model and interpretation in context. In terms of overarching themes, this covers OT3.1 (though not including making assumptions), OT3.2 and OT3.3. Some variations are given below that develop the question to cover more of the overarching themes and assessment objectives.

Variations

1. OT3.1 – making simplifying assumptions. In the original question the key assumptions have been made in the stem, that is, that the balloon is spherical and the proportionality statement. Either of these is difficult to get candidates to suggest within this question because they have a bearing on the nature of the differential equation produced and therefore on the analysis.

However, it seems reasonable to ask candidates to set up such a model in isolation, potentially even to take it as far as setting up the differential equation but no further. They would need to write down a list that covered:

- the shape of the balloon, including variables to represent the relevant dimensions;
- a rate of change of a given dimension with respect to time in terms of either time or that dimension;
- initial or boundary conditions

and then translate that into a differential equation. This is not a style of question that is currently used, and it would be challenging for candidates.

2. OT3.4 and OT3.5 – understanding and using assumptions, refining models. There are plenty of potential alternatives to part (ii), or additional parts that would cover these overarching themes, for example:

- Consideration of the long-term behaviour of the model and suggesting a refinement that would improve the model – for example setting a range for t that fits with “normal” balloons, setting a volume that you want the balloon to reach and setting a range for t that fits.
- Change the model, for example modelling the balloon as a cylinder, and ask candidates to consider how this model might be set up and what would need considering – for example, that one might want to set the radius as constant, or as a constant proportion of the length.
- Ask candidates to adapt the model so that after a given amount of time air is released from the balloon.
- Ask why modelling the rate of change of the radius as inversely proportional to $t - 3$ would not be sensible.

3.3 Implications for mark schemes

While the full modelling cycle need not be assessed in its entirety within a single task, candidates are required at some point to construct their own models. When a number of models are possible or feasible, the range of models constructed by candidates could be wide. If the candidate is subsequently required to work with their model, the complexity of the candidate's model will have an effect on the complexity of the remaining parts of the task. There is an issue here of how to reward simpler (and potentially but not necessarily less effective) models against those which might be more ambitious and complex (and potentially but not necessarily more effective). In other words, mark schemes must be able to reward candidates consistently and reliably, regardless of which model they choose. However, it is important to ensure that this is kept within the bounds of reasonable assessment and it would be inappropriate, at this level, to ask questions that were so unstructured that any approach to modelling would be acceptable.

The potential solutions for this issue are similar to those discussed above in relation to problem solving questions with multiple valid approaches. Multiple mark schemes covering possible models or levels-based mark schemes could be useful here but, as with problem solving, these come with some risk and could, in practice, affect the achievement of assessment objective weightings.

4 Large data sets in statistics

4.1 Detailed content expectations

The subject content for A level mathematics requires candidates to be familiar with one or more specific large data sets, to use technology to explore the data set(s) and associated contexts, to interpret real data presented in summary or graphical form, and to use data to investigate questions arising in real contexts. This requirement reflects a desire to change the way in which statistics is taught, and this has implications for assessment. As such, it is important to focus on how this could be achieved and the nature of the data sets that candidates should be working with, rather than specific areas of assessment. To this end, a detailed description of the characteristics desirable in the data sets to be studied has been provided below as this could be of more use than exemplars.

The subject content also includes other aspects of statistics to be assessed and it is the intention (although not a requirement) that this content be delivered through the context of large data sets where appropriate. Assessment of large data sets themselves in the context of examinations is challenging as there are limitations on the time and resources available to candidates. With that in mind, as well as covering the detailed statistics content, examinations should, as far as possible, assess familiarity with the specified large data set(s).

As the content expectations for large data sets relate principally to the teaching and learning of statistics, a signed statement, similar to that required for GCSE, AS and A level geography fieldwork, could be used to ensure that centres use large data sets in the teaching of the statistics content, and this could include details of the exact data sets used. While this might be desirable, it could impact on, for example, arrangements for private candidates and this must be carefully managed.

It is essential that exam boards consider carefully the data sets chosen for study and how they are used, how often they should change, how many should be studied and how (if at all) they should be used in an examination. If a data set is too small or does not regularly change, this could lead to predictability of questions and to teaching to the exam, thereby having a negative impact on classroom behaviour and not securing the curriculum intentions set out in the subject content.

4.2 Characteristics of large data sets and their use

Desirable characteristics of large data sets for AS and A level mathematics

- The data set(s) should consist of real data and, where possible, the source should be given (including URLs) so that candidates can find out how the data set was collected. Data may be reorganised into a standard format but the data should not be cleansed.
- The data set(s) should be large enough for exam boards to manage the risk of predictability, whilst being manageable in the classroom and capable of supporting the teaching of the statistics sections of the prescribed content.
- The large data set(s) should contain a mixture of categorical and numerical data.
- Cleansing data will arise naturally for missing data values and in dealing with possible errors. Exam boards should consider the differing data-quality issues that arise with primary and secondary data, and the treatment of errors and outliers.
- One of the purposes of the data set(s) is to ensure that candidates are familiar with the terminology and contexts relating to the data. This allows tasks to be set in real contexts without requiring lengthy explanations of the terminology and contexts. It also enables candidates to think about the data in advance and so be ready to engage in realistic interpretation.
- A short piece of text and/or a glossary may accompany the data set(s) to help candidates understand the source of the data and associated terminology, but the requirements of the assessment should ensure that candidates have had a greater engagement with the data than this text/glossary would provide.
- The data set(s) should be suitable for analysis using a spreadsheet or statistical data package.
- Candidates would not be required to have a printout of the whole data set(s) in the examination, but selected data or summary statistics from the data set(s) can be provided either as a handout to select from or within examination tasks.
- It should be clear to candidates whether the whole data set is (essentially) a population or whether it is a sample from a larger population.
- Working with a sample and making an inference about a population is an important feature of statistics – if the data set is a whole population then samples associated with the data set may be pre-released alongside it or introduced in examination tasks.
- The skills candidates will learn by working with the large data set(s) are general and applicable to a variety of data sets. However, the data set(s) used should support examination tasks that are not predictable.
- Tasks may be based on samples related to the contexts in the pre-release data set(s); their work with the pre-release data set(s) will help candidates understand the background context.

- The familiarity candidates gain with contexts related to the large data set(s) will enable them to answer questions about interpretation of data that are often found difficult by candidates when the contexts are unfamiliar.

4.3 Implications for mark schemes

To fully realise the subject content requirements in relation to large data sets, tasks must be designed in a way that provides greater opportunities to score marks to those candidates who can demonstrate familiarity with one or more large data sets. Similarly, mark schemes must specifically reward instances where this familiarity is demonstrated and avoid providing equal reward to responses that can be drawn from the question itself or can be reasonably inferred by using common knowledge or knowledge of large data sets in general.

5 Conclusion

The working group has provided a rich source of evidence on mathematical problem solving, modelling and the use of large data sets in statistics and provided examples and descriptions of assessing these in practice. In doing so, the group recognises the tensions that exist between the delivery of curriculum aspirations and the delivery of reliable assessments on a national scale.

While agreeing on a definitive list of problem solving attributes is challenging and could constrain the setting of high-quality questions if adhered to rigidly, the list in section 2 demonstrates there is agreement on what is meant by problem solving within an examination context, and provides some common ground on which assessment developers can begin to design AS/A level problem solving tasks. The group was of the view that there may be instances in which problem solving skills or attributes are demonstrated within questions that focus on other aspects of the content, but that candidates must be given opportunities to attempt to complete unstructured problems that require the use of multiple parts of the problem solving cycle as described in the overarching themes. The working group also identified the need for mark schemes to be able to reliably and consistently reward candidates for demonstrating problem solving skills, regardless of the method chosen, and has suggested some possible options for exam boards to consider.

In terms of modelling, the group has found that the current approach to assessment remains appropriate but must be extended across more of the modelling cycle, and across more of the content as identified in the detailed content statements. The same issue with mark schemes as identified for problem solving was raised in relation to modelling. Where candidates are required to construct their own model and use it to answer a question, mark schemes must be able to deal with different models being used, which could affect the complexity of the subsequent mathematical working. Mark schemes will need to give significant reward for the suitability of the model chosen by a candidate in response to a task.

For the use of large data sets in statistics, the group has highlighted the desire to change the way in which statistics is taught to incorporate the use of large data sets, and much of the discussion was around how exam boards could influence this through assessments and the challenges associated with this. All members agreed on the need for a substantial program of continuing professional development for teachers if this intention was to be fully realised.

While much of this report provides information relevant to exam boards as they develop assessment materials, as per the remit of the group it also provides evidence for Ofqual to consider. These findings will be considered when finalising assessment objectives and their weightings and in preparing regulatory documents such as Conditions, Requirements and guidance.

A key policy aim of A level reform is to maintain the current A level standard. There is potential for the combination of the new content, which is very challenging in some areas, and problem solving, together with the move to a linear qualification, to raise the level of demand beyond what is appropriate. This will be a key consideration when finalising the assessment objectives, as these will need to be balanced in a way that ensures problem solving is central to AS/A level mathematics and further mathematics as required by the subject content but does not inappropriately raise the level of demand.

For further mathematics, many of the findings of the working group are relevant, particularly the implications for mark schemes where multiple approaches may be validly used to answer a question. In further mathematics there is significant flexibility in the content to be included in specifications. We must therefore develop regulations which allow that flexibility to be exercised without negatively affecting comparability of standards across specifications or across multiple routes within one specification.

Appendix 1 – A Level Mathematics Working Group, Terms of Reference

A Level Mathematics Working Group

Terms of Reference

1 Role

1.1 The role of the working group is to provide a source of evidence to support Ofqual in making regulatory decisions for AS and A level mathematics and Further Mathematics for first teaching in 2017.

2 Our proposals

2.1 Specific areas of the working group remit are described in section 4.

2.2 The output of the working group will be an informal report that will be used as one of the sources of evidence for Ofqual to consider in providing any guidance and making regulatory decisions. Neither the working group nor any records of discussions or reports will have regulatory status.

2.3 Ofqual will use the evidence generated by the working group, in the form of the informal report and other working group material, alongside other evidence, to:

- support the finalising of assessment objectives and weightings for these subjects;
- inform the development of the technical interpretations of the assessment objectives;
- support the development of high-quality assessments, particularly in relation to mathematical problem solving.

3 Membership

3.1 Ofqual is responsible for establishing the working group, for determining and revising its structure, for approving and updating its terms of reference, and for disbanding it.

3.2 The membership of the working group consists of:

- a senior member of Ofqual staff, who will chair meetings;
- two members of the mathematics panel convened by ALCAB, who will be invited directly;
- two members proposed by mathematics associations;
- two representatives nominated by each exam board offering these qualifications, comprising one mathematics specialist and one representative with sufficient authority to agree consensus views on behalf of their respective boards;
- two practising teachers of A level mathematics;
- a maximum of three Ofqual subject experts for these qualifications.

3.3 Members from the panel convened by ALCAB and members proposed by mathematics associations will contribute in an individual capacity and not as a representative of any organisations, associations or groups to which they may belong.

3.4 Secretariat for the working group shall be provided by Ofqual.

4 Responsibilities

4.1 The working group will carry out its responsibilities between March and July 2015.

4.2 The responsibilities of the working group are, within the scope of the published subject content and decisions already made about assessment structures, to:

a) produce a range of exemplar examination questions and marking principles suitable for these qualifications;

b) identify instances of good practice or innovation within these exemplars;

c) produce supporting commentary on the exemplars.

4.3 These questions will focus primarily on mathematical problem solving and will cover the mathematical modelling process and statistics in relation to large data sets.

4.4 The evidence generated by the working group must include explanations, in the exemplar questions identified and marking principles proposed, of how:

- content expectations are met;
- questions have been set at an appropriate level of mathematical demand (and how the questions address the ability range);[9]
- progression has been achieved from requirements of problem solving at GCSE.

4.5 The working group will produce a report, detailing its findings in relation to the role and responsibilities described in sections 1 and 4.

4.6 Members will nominate a member(s) to lead on the development of the report.

5 Meetings

5.1 The working group is expected to meet at least twice, once in March 2015 and again in early summer 2015. The Chair of the working group may call additional meetings as required to ensure that the responsibilities of the working group are met.

5.2 Ofqual will aim to send a draft agenda to members at least 10 working days before the meeting. All members will be given an opportunity to propose items of business for inclusion on the agenda. Requests for agenda items and/or papers from either members of the working group or others should be provided to Ofqual at least eight working days before the meeting. The final decision for inclusion rests with the Chair.

5.3 Notes of key discussion points and actions will be produced by Ofqual. Ofqual will aim to circulate these within five working days of the conclusion of each meeting.

[9] As part of the explanations of how the exemplars are set at an appropriate level of mathematical demand, the group should give consideration to the grades at which a question is targeted and, subsequently, how such questions might feature in the context of full question papers.

5.4 Meetings will be the key forum in which the working group will carry out its duties, but there is likely to be communication required between members outside of meetings.

5.5 Discussions outside of meetings are likely to be required in order to produce and finalise the report.

6 Decision making

6.1 The working group is not a decision making body. Its role is to produce a report containing information, advice and recommendations as described in section 4.

6.2 To the extent that the working group needs to be able to reach agreement on matters in the performance of its functions, agreements will be achieved on the basis of general agreement.

6.3 Areas in which agreement is not achieved can be described in the final report.

7 Resources

7.1 Any costs (other than travel to meetings convened by Ofqual) associated with membership of the working group are to be covered by the relevant party. Resources will not be provided by Ofqual.

7.2 Ofqual will cover travel costs for members to attend meetings of the working group, the cost of meeting room hire and refreshments during meetings.

8 Review

8.1 Once the working group produces its report in July 2015, it will be disbanded.

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