Key Stage 3

National Strategy

Guidance

Curriculum and Standards

Interacting with mathematics in Key Stage 3

Constructing and solving linear equations

Pre-course reader

Teachers of mathematics

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Introduction

This reader is preparation for the first session of the algebra course *Constructing and solving linear equations*. It includes an extract from *Key aspects of teaching algebra in schools* (QCA/02/913) by John Mason (Open University) and Rosamund Sutherland (University of Bristol).

Activity

As you read the extract, briefly note or highlight Mason and Sutherland's view of why we teach algebra. How does this compare with your own view?

Acknowledgements

Thanks are due to QCA for permission to reproduce extracts from *Key aspects of teaching algebra in schools* (QCA/02/913) by John Mason (Open University) and Rosamund Sutherland (University of Bristol)

Extract from *Key aspects of teaching algebra in schools*

Why algebra?

All of the summaries and many of the papers we reviewed agree, implicitly or explicitly, that algebraic thinking contributes to being a full citizen able to participate fully in the democratic process, and that algebra is the language in which the use of mathematics in economic activity is expressed.

From a democratic point of view, any citizen who is unconfident with expression and manipulation of generality cannot function fully in the political and economic process, because modern society runs on the assertion and critique of generality, including the use of mathematical models to study and predict the effects of policy decisions. Citizens unable to engage in this debate are disenfranchised.

In an industrial culture, owners, factors, and managers all need to deal with the general in formulating (note the etymology) and deciding amongst different policies and when determining procedures to be followed by employees, which is in essence, a form of algebra. By contrast, customers are interested only in the particular application of these rules to their situation. However, citizens need to be able to engage in thinking about the general in order to appreciate how those decisions are being made.

In a knowledge-economy, everyone who participates is faced with assertions of generality concerning policy decisions and choices. Citizens need to be able to analyse and critique these assertions and the models which underlie them, and to assert their own versions. Algebra provides the basis, the language, the foundation for this participation.

Today's society places considerable emphasis on the use of technological tools such as spreadsheets and databases. These have their roots in the early development of computer programming languages, which in their turn have their roots in mathematics generally and algebra in particular. Thus, it can be argued that today's citizens should both appreciate and become competent in the generalising and symbolising power of algebra, in order to be able to understand the potential and the constraints of these computational packages. Software only does what it has been 'programmed' to do.

Abstraction as strength and as weakness

'Abstraction from context', which Diophantos achieved for early algebra nearly 2000 years ago, is a source of the power of mathematics, for abstraction enables:

- concentration on the central technical problem to be solved independent of the particular context in which it is embedded;
- further and deeper study of more general structures in a search for an effective solution to a class of problems; and
- application of those techniques in a variety of superficially very different contexts.

Unfortunately, this very strength is a weakness when it comes to education, for there is a strong temptation to teach the abstracted technique isolated from all context, and a converse temptation to teach the technique as a set of rules to be followed in specific contexts. Neither has proved successful on its own, hence the tension between, for example, modelling and word-problems as approaches to the introduction of algebra.

Interweaving of research and educational perspective

Reviewing literature from around the world reveals subtle but important differences in approaches to research, to mathematics, to teaching, and in particular to algebra, and these differences must be borne in mind when seeking to construct a curriculum that 'works' in the context of England and Wales.

Principal issue

The biggest issue is not 'how best to teach algebra', because any programme of materials and tests is likely to degenerate, as several authors suggest in one way or another, into the mechanical and the routine: the *transposition didactique* formulated by Chevallard (1994, 1985) in which expert awareness is transformed into instruction in behaviour. In other words, the richness of the expert's connections and competencies, when turned into teaching materials, becomes a collection of behaviours for students to mimic and master. Algebra teaching has always been particularly prone to degenerate from expressing generality into manipulation of letters as if they were numbers. For example, despite the avowed desire for students to learn to use letters to express general relationships, books of exercises such as Humphreys (1938), which is typical of the problems posed to students over more than a century, are reduced to using letters as if they were numbers with no sense or hint that there might be a generality present. This is evidenced by the lack of stimulus to generalise those parts of word-problems which use particular numbers.

The issue, therefore, is how to strengthen and develop awareness and appreciation of the various aspects of algebra in every topic, amongst both serving and newly-qualified teachers. Every topic, every lesson, offers opportunities for using and extending algebraic thinking, and unless algebraic thinking imbues teachers' ways of preparing for and conducting lessons, algebra will continue to be the principal mathematical watershed for most people.

What is vital is that teachers use their own awareness of the centrality of algebra in mathematics as a computationally expressive language, to inform their practice in every lesson, not just in lessons labelled 'algebra'. This requires teachers to be encouraged to develop their own awareness. Working on awareness is not a one-shot event, but a career-long enterprise. For example, departments in which teachers work together on mathematics new to them are better placed to refresh their awareness of what students experience and to refresh their awareness of the roles of algebra than are departments in which teachers do not develop their own mathematical thinking.

Kaput (1999) neatly summarises the demands of teaching algebra in the 21st century:

> Begin early; integrate algebra with other subject matter; include several different forms of algebraic thinking (problems, modelling, generalising, functional thinking); build on children's natural linguistic and cognitive powers; encourage them to reflect upon and become aware of those powers so that they learn to articulate what they know; encourage children to make (mathematical) sense of the world around them and of what they are taught.

Arcavi (1994) puts it more succinctly:

> Algebraic symbolism should be introduced from the very beginning in situations in which students can appreciate how empowering symbols can be in expressing generalities and justifications of arithmetical phenomena ... in tasks of this nature, manipulations are at the service of structure and meanings. (p. 33)

Approaches based on manipulables (Sawyer, 1959), on Babylonian area diagrams and on the balance metaphor (Filloy and Sutherland, 1997) or on algeblocks … or polynomial engineering (Simmt and Kieren, 1999), while achieving some success in the short-term, face the problem of weaning students off the use of material objects and onto the use of mental objects, and further, onto the use of symbols to denote these objects. Seeking solutions in digital technology alone is dangerous, for although software enables students to get a machine to manipulate algebra for them, they need at least some experience in that manipulation in order to know what to ask the software to do for them. Exactly how much and of what form requires further research.

Distinctions and dualities

The following distinctions arise in many of the reviews and reports, expressed in a different language and with different emphases. In our view, they all need to be taken account of in the design of a curriculum and in the description of pedagogical practices, as well as in future research.

Object-process

An expression such as $3x + 4$ is both the answer to a question, that is, an object in itself, and also an algorithm or process for calculating a particular number. Being aware of this has been called *proceptual thinking* (Gray and Tall, 1994). Arithmetic, including arithmetic with symbols, places an emphasis on the process of calculation and thus many students are not aware of this duality. Absence of this dual perception accounts for many of the classic errors with symbols observed being made by students who only experience arithmetic with letters.

Analysis-synthesis (arithmetical-algebraic)

In arithmetic one proceeds from given, known numbers to calculate as-yet-unknown numbers, arriving eventually at a final answer (what Viète, 1591, called *the analytic art*). In algebra one proceeds by denoting what is not-yet-known ('acknowledging ignorance', Mary Boole in Tahta (1972, p. 55), expressing calculations on those as-yet-unknowns to produce constraints, then seeking solutions to those constraints and re-interpreting them as solutions to the original problem. This is also typical of modelling more generally, for the algebra is being used to model-express the situation and the constraints.

There are considerable initial psychological differences between moving from confidence into the unknown, to starting with the unknown and calculating 'as if' it were known. It is a reasonable conjecture that once letters become a familiar vocabulary in which to express generality and constraints, these psychological differences are likely to disappear.

Unknown-general-variable-parameter

Letters are used in four different yet inter-related ways:

- to denote a specific unknown whose values are sought (what is the scope of generality given the constraints imposed?);
- to denote a general or unspecified number which can take any one of a range of values; all expressions using that symbol are either valid, leading to the notion of an identity, or are constraints, leading to the notion of equations and inequalities;
- to denote a quantity which is permitted to vary over a specified range (variable), used particularly to study the properties of functions; and
- to denote a quantity which could be allowed to alter but which for the moment is considered to be fixed (parameter); arises especially when generalising in the context of the study of something else as variable.

Structural-empirical

One example, seen generically or paradigmatically, that is used to see through to the general, can give access to experience of structure in a situation, problem, etc. Several, even many, examples can be used empirically to locate and express a pattern (guess a formula). Empirically abduced or induced formulae need to be justified by recourse to the source of the numbers; structurally deduced formulae need to be justified by articulating the identified structure.

For example, an empirical approach to a sequence or set of numbers is to analyse them using finite differences or using a statistical technique such as linear regression to locate a possible formula which generates all the known cases, perhaps approximately; a structural means of building each term from preceding terms can be identified and expressed, and then the recurrence relation can be used to try to generate a formula;

the source of the sequence of numbers can be examined and analysed to reveal a structural formula for the general term in the sequence.

Empirical pattern spotting is often a matter of 'going with the grain', whereas the important structural awareness emerges by 'going across the grain' (Watson, 2000), but there is more to it than mere 'trainspotting' (Hewitt, 1992).

Proof and problem-solving

Mathematics is seen by many as being as much about proof as it is about problem-solving, although trying to convince others can in fact be seen as a problem in itself! Proving, or justifying, or reasoning, *or convincing yourself and others*, is a process which depends upon a symbol system for representing the objects about which something is to be proved. Reasoning then proceeds by expressing relationships or necessary consequences. Proof necessarily involves reasoning with generalities, showing that any and every case will conform with the justification offered.

Whereas empirical approaches can be taken in finite situations where all possible cases can be listed, addressed or tested, once there are infinitely many possibilities, some sort of language is needed in which to express the general. For example, the fact that the sum of two odd numbers is even and their product is odd is just an initial step on the road to studying the difference between conjectures based on particular examples, and certainty based on assumptions and reasoning, certainty over an infinite class of cases. An early example which many children construct for themselves is that there is no largest number ('I can always add one to anything you say'). It is quintessential mathematical reasoning, expressing a generalisation of an action performed in several particulars, and imagined as possible in any such situation.

Themes

In much of the writing reviewed there are both traces of, and direct references to, major themes which pervade mathematics and which serve to link and unify apparently disparate topics through the approach taken or through underlying structure which emerges. Here we mention briefly seven:

Mental imagery

Expressing oneself in succinctly manipulable symbols involves the use of mental imagery as a mediator between the situation (as imagined) and the situation (as abstracted and symbolised). Similarly, manipulation of symbols involves anticipation of what will be achieved and of what form is sought (Boero, 2001). This is another important role for mental imagery.

Freedom and constraint

Most algebra problems can be seen as starting with a free choice of number, expression or function, and then imposing constraints on the choice, leading to the problem of determining whether there are any numbers, expressions, or functions which satisfy those constraints, and how to identify those that do.

Invariance amidst change

Most mathematical results are statements about something which remains invariant while other things are permitted to change. Stress is usually placed on the invariant, but in order to appreciate it, it is necessary to be aware of the scope of permitted variation or change. For example, the sum of the angles of a planar triangle is 180° which states that the angle-sum is invariant, but obscures awareness that this is true for *any planar triangle whatsoever*. Students often do not appreciate the import of the generality because they are unaware of the range of change within which the generality remains valid. Explicitly varying elements is often necessary if students are to learn that dimension of variation is possible within the concept (Marton and Booth, 1997).

Doing and undoing

Whenever a calculation is performed to reach an answer, it is possible to reverse the process and to ask, could this (another expression, number etc.) have been a possible answer, and if so, to ask what the corresponding question was. At an elementary level, this is the structure which produces a need for negatives (what number could be added to 5 to give 3?) for rationals, and later for complex numbers among others.

It is also a device for producing challenging and creative problems. For example, is there a configuration in the game of jumping-pegs or leapfrogs which would take exactly 29 moves? Are there entries at the vertices of an arithmogon to give specified values along the edges?

Doing and undoing often leads to characterising those numbers or expressions which could be answers, and distinguishing them from those that could not.

Characterising and organising

Much of mathematics concerns characterising objects, such as the kinds of numbers which can arise as the solution to a specific problem (e.g. 'one more than the product of four consecutive integers', or 'cannot be factored'), often in association with undoing or reversing a calculation process.

Extending meaning

Throughout school, students meet the same words used in contexts which include but extend their old use. Thus, numbers start as 'counting' or 'whole' numbers, then include the negatives, the rationals (strictly speaking, fractions are not numbers but ratios, and become numbers when all the ones with the same value are identified), numbers of the form $a + b\sqrt{n}$ for some fixed n where a and b are rationals, the complex numbers. Rational polynomials are number-like but curiously *not* considered to be numbers. At each stage, the familiar numbers are extended by demanding that the arithmetic remains consistent. Something similar happens when trigonometric ratios (sine, cosine, etc.) are replaced by power series, solutions to differential equations, etc.

The language of generality

In English, the words *a* and *any* can be used to indicate a generality (as can *all* and *every*), but can sometimes be used confusingly to indicate a particular:

- Consider *a* number: is it particular or general?
- The sum of the angles of *a* triangle (particular or general);
- Take *any* number between 1 and 10 (is attention on the choice of one or on the fact that it can be any?).

This can be confusing to students, especially those for whom English is not their first language.

Similarities and differences

A quick reading of the literature suggests that there are several different approaches to introducing algebra in school. For example, Bednarz *et al.* (1996) is structured around modelling, problem-based, generalisation-based, and function-based approaches and similar distinctions are articulated in different papers which emphasise one aspect or the other. But in the final analysis, is there a significant difference between the different approaches distinguished? Is there a significant element added by the use of calculators and software?

It is certainly possible to emphasise differences, as authors are prone to do. But it is also possible to see great similarities in essence despite differences in rhetoric and discourse. They all involve taking some situation, whether it arises in the material world or in some imagined or mathematical world, forming a mental image of the essence of the situation and of the relationships involved, expressing these in the language which at school is called algebra, manipulating the symbols so as to resolve the mathematical problems which emerge (solving equations or inequalities, isolating certain variables, finding integer solutions, etc.), and finally, testing these solutions against the original situation to check for appropriateness.

Differences in pedagogy arise when the manipulations are isolated and emphasised at the expense of expression, so that students are faced with rules and techniques without participating in construction and communication of meaning. This returns us to the opening theme of this section and to the virtually universal agreement amongst authors, that to be effective, the teaching of algebra has to engage students in constructing and communicating meaning, and that manipulation is a by-product not the focus or purpose of teaching algebra.

In conclusion

Imbuing every lesson with algebraic thinking, with expressing generality and particularising generalities, with conjecturing and reasoning, is vital to successful experiences with algebra. All dimensions of algebraic manifestations mentioned here (see following subsections) must be intertwined so that students can develop and use their undoubted powers to think (and to enjoy thinking) mathematically through the medium of algebra.

> Algebra is now [1986] not merely 'giving meaning to the symbols' but another level beyond that: concerning itself with those modes of thought that are essentially algebraic – for example, handling the as-yet-unknown, inverting and reversing operations, seeing the general in the particular, [imposing constraint on freedom]. Becoming aware of these processes and in control of them, is what it means to think algebraically.

(Love, 1986, p. 49, quoted in Wheeler, 1989, p. 282, square brackets added)

Source: *Key aspects of teaching algebra in schools* (QCA/02/913) by John Mason (Open University) and Rosamund Sutherland (University of Bristol), pages 6–12.

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