Interacting with mathematics in Key Stage 3

Year 9 proportional reasoning: mini-pack

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Year 9 proportional reasoning: sample unit

Introduction

This unit is a sequel to the Year 8 multiplicative relationships unit. It provides an opportunity to revise, consolidate and extend ideas introduced in Year 8 and to make links to other mathematical strands, particularly shape and space. Making such links, especially with visual contexts, can help pupils to understand proportion. There is perhaps no other single concept in secondary mathematics which is so important and so pervasive. Drawing attention to multiplicative relationships as they arise in other topics provides opportunities to assess pupils' understanding of the key ideas and their ability to use and apply these. For some pupils this unit will represent a second opportunity to come to terms with the big ideas involved in multiplicative thinking. Other pupils may have been using and applying strategies developed from the Year 8 unit and will be ready to refocus on proportional reasoning and develop more sophisticated methods for solving problems.

This Year 9 unit has been developed through a flexible use of the *Sample medium-term plans for mathematics*.

In planning the unit several decisions were made that affect the medium-term plans for mathematics.

- The objectives for the unit are mainly drawn from Number 1, with some from Number 2 and Shape, space and measures 3.
- It is taught during the autumn term of Year 9 and replaces Number 1.
- Some objectives concerning use of the calculator are covered here and will need less time in Number 2.
- Objectives on calculation from the existing Number 1 will need to be covered in Number 2.

Proportion problems are sometimes solved quite easily using an informal approach, particularly when the numbers involved are simple multiples of each other, as in a recipe conversion. But pupils also need to be able to deal with the general case; this requires a higher level of understanding and problem-solving skills, using the operations of multiplication and division. A key feature of the oral and mental starters in this unit is the automating of some of the processing tasks. Throughout the unit, calculators remove the burden of arithmetic, which helps pupils to focus on the relationships involved.

This unit has been structured into three phases for teaching.

Phase 1 (about three lessons)

- Phase 1 builds on the idea of scaling numbers, introduced in the Year 8 mini-pack, to consider successive scalings and their inverses.
- It re-examines the use of tabular arrays to set out proportional relationships and find unknowns (followed through in phases 2 and 3).

Phase 2 (about three lessons)

- Phase 2 links numerical work to geometry – the invariant properties of enlargement and the concept of similarity.
- It considers relationships 'within' and 'between' similar shapes.

Phase 3 (about three lessons)

- Phase 3 considers problems from a variety of contexts (numerical, graphical, spatial, statistical), to identify the underlying relationships of proportionality.
- It revises and develops methods for solving proportion problems.
- The phase places particular emphasis on problems involving percentage increases and decreases.

Objectives

A Understand the effects of multiplying and dividing by numbers between 0 and 1; use the laws of arithmetic and inverse operations; *recognise and use reciprocals.*

B Enter numbers into a calculator and interpret the display in context (negative numbers, fractions, decimals, percentages, money, metric measures, time).

C Recognise when fractions or percentages are needed to compare proportions; solve problems involving percentage changes.

D **Use proportional reasoning to solve a problem, choosing the correct numbers to take as 100%, or as a whole;** *understand and use proportionality and calculate the result of any proportional change using multiplicative methods; understand the implications of enlargement for area and volume;* compare two ratios; interpret and use ratio in a range of contexts, including solving word problems.

E Enlarge 2-D shapes; *recognise the similarity of the resulting shapes; identify the scale factor of an enlargement as the ratio of the lengths of any two corresponding line segments; recognise that enlargements preserve angle but not length.*

F Solve increasingly demanding problems and evaluate solutions; explore connections in mathematics across a range of contexts: number, algebra, shape, space and measures and handling data; *generate fuller solutions.*

G Represent problems and synthesise information in algebraic, geometric or graphical form; move from one form to another to gain a different perspective on the problem.

H Suggest extensions to problems, conjecture and generalise; identify exceptional cases or counter-examples, explaining why; *justify generalisations, arguments or solutions; pose extra constraints and investigate whether particular cases can be generalised further.*

Differentiation

Ideas for progression in oral and mental starters are provided in a set of prompts.

Support in the first phase could be provided by devoting more time to revising ideas developed in the *Year 8 multiplicative relationships: mini-pack*.

The unit refers to the Framework supplement of examples, particularly in phase 3 of the unit. In the case of each objective the pitch of the work is accessible through Years 7 to 9 (and Year 9 extension) and examples can be chosen appropriately.

Further ideas for varying the level of challenge are detailed in the prompts for main activities.

For some pupils, it would be appropriate to revisit these concepts through lessons 3, 5 and 12 in the *Year 9 booster kit: mathematics*.

Resources

- Calculators, scissors
- Paper:
  - two large square sheets, one smaller square sheet (class demonstration)
  - two sheets each of A4 and trimmed A3 paper for folding and cutting (per pair of pupils)
- Selected sets of proportion problems, drawn from the Year 8 problem bank and other suitable sources (see 'Prompts for main activities in phase 3' page 25)
- Resource sheets – to prepare as OHTs and/or handouts (included in the school file and on the CD-ROM):
  - Scaling line segments (2 sheets)
  - Cat faces
  - Photographic enlargements
  - Shadows (4 sheets)
  - Four problems: making the links
- Supplementary notes (pages 10–28 of the mini-pack):
  - Prompts for oral and mental starters: phase 1
  - Prompts for oral and mental starters: phase 2
  - Prompts for oral and mental starters: phase 3
  - Prompts for main activities in phase 1
  - Prompts for main activities in phase 2
  - Prompts for main activities in phase 3
  - Prompts for final plenary in phase 2

Key mathematical terms and notation

scaling, scale factor, multiplier

inverse operation, inverse multiplier

reciprocal

ratio: expressed using ratio notation ($a : b$) or in equivalent fraction ($\frac{a}{b}$), decimal or percentage forms

proportion:

- as a fraction or part of a whole ('proportion of')
- as equality of ratios, i.e. direct proportion ('in proportion', 'proportional to')

rate, per, for every, in every

unitary method

enlarge, enlargement, centre of enlargement

similar, similarity

Unit plan

Oral and mental starter	Main teaching	Notes	Plenary
Objectives A, B **Higher/lower** Give fraction multipliers. Pupils have to say whether the result will be higher or lower. Introduce the term *reciprocal* and include unitary fractions. **About** Give multipliers (fractions, then decimals) to apply to a specified number. Pupils have to approximate the size of the result. **Calculator quick** Give a calculation (e.g. $\frac{3}{7} \times 5$). Pupils use calculators and discuss key sequence.	**Phase 1 (three lessons)** Objectives A, D, H **Repeated scaling** Introduce repeated scaling (e.g. $4 \rightarrow 7 \rightarrow 5$) using multiplication only, representing numbers by line segments. Recalling work from the Year 8 multiplicative relationships unit, identify scale factors ($\frac{7}{4}$ and $\frac{5}{7}$), the equivalent single scale factor ($\frac{5}{4}$) and inverses. Give a reminder of the terms *scale factor*, *multiplier*, *inverse multiplier*. Check that decimal scale factors give consistent answers. Discuss other examples, working towards the general result $\frac{b}{a} \times \frac{c}{b} = \frac{c}{a}$. **Proportional sets** Referring to the plenary of the previous lesson, remind pupils of the terms *ratio* and *in proportion*. Give a table with three columns and up to ten rows of numbers in proportion. Leave some entries blank. The task is to find these entries using relationships between rows or columns. Pupils have to select appropriate 2 × 2 arrays to calculate each unknown, setting out tabular extracts and equations. Discuss alternative approaches and mix examples suitable for mental and calculator methods. Set A Set B Set C 2 7 x 2 x $x = 2 \times \frac{7.5}{5}$ or $x = 7.5 \times \frac{2}{5}$ 5 y 7.5 5 7.5 z 28 t . . .	**Support:** See Year 8 multiplicative relationships unit. Scalings of line segments can be expansions or reductions. Mathematically, these are all *enlargements*, sometimes with scale factors less than 1. At a suitable point, discuss links between scale factors using fraction and ratio notation, e.g. if $a : b : c = 2 : 7 : 3$ the scale factors between them are $\frac{7}{2}$, $\frac{3}{7}$, $\frac{3}{2}$, etc. **Extension:** Consider how to place data so that missing entries can be found (last page of prompt sheet).	Generate three proportional sets, given specified ratios, e.g. $a : b = 3 : 5$ and $b : c = 10 : 3$. Start by choosing entries for one set. (Could be set as homework.) For a given unknown, pupils offer several different tabular extracts from which it can be calculated. Discuss efficient choices.
Objectives C, D **Equivalent lists** Give a ratio of two quantities. Pupils give scale factor as a fraction and a decimal, and the percentage increase or decrease. Same for inverse and other ratios. **Webs** For example, if £5 = €8 (5:8), give some more equivalents. Set out as web diagram and include entries for £1 and €1.	**Phase 2 (three lessons)** Objectives D, E, F, G, H (Framework pp. 193, 213–215) **Folding paper** Teacher demonstration: Start with a square, fold along a shorter line of symmetry and cut in half. Repeat four more times. Arrange rectangles and discuss, noting two sets with the same proportions (1 : 1 and 2 : 1). Class exercise (in pairs): repeat for rectangles, half the class starting with a metric sheet (e.g. A4), half with a non-metric sheet. Arrange rectangles, tabulate lengths and widths, find scale factors/ratios. **Cat faces (OHT and resource sheet)** Which faces are similar and how do you know? Consider: • ratios of features within a face that are preserved in a similar face; • multipliers (as fraction, decimal and percentage) that scale between similar faces.	Emphasise tabulation of data and make links with phase 1 of the unit. Define mathematical *similarity*: dimensions in proportion – ‘same shape, different size’. Note that all squares are similar. **Extension:** For square folding use visualisation and discuss. Include area scale factors.	Discuss results from metric and non-metric rectangles, noting that, for the former, all rectangles have the same proportions. Choose plenary depending on progress with main activities. For example: • As well as squares and circles, what other shapes are always similar? • Raise a suitable thinking point relating to photographic enlargements (prompt sheet).

Oral and mental starter	Main teaching	Notes	Plenary
'Splits' For example, divide £36 in different ratios (e.g. 1 : 5, 5 : 7, 1 : 2 : 3, …). Set out as web diagram.	**Photographic enlargements (OHT or resource sheet)** Given three similar photographs, discuss what values need to be given to determine six dimensions (lengths and widths), six scalings between photographs, and six internal ratios between lengths and widths.	**Support:** Guide pupils, revealing more information as needed. **Extension:** Concentrate on general principles about what values might be given.	**Extended plenary** OHT Shadows 1: Given lengths of shadows and height of child, find heights of trees. OHT Shadows 2: Given length of child's shadow later in the day, find lengths of shadows of trees.
Objectives A, B, C, D **Working towards fluency** (Extend to up to 15 minutes) • Using calculators with 'awkward' numbers • Rapid conversion between ratio, fraction, decimal and percentage forms • Numbers and quantities, using rates (clearly stated '… per …') Cover these calculations: • Expressing proportions: $\frac{a}{b}$ • Finding proportions of …: $\frac{a}{b}$ of … • Comparing proportions: $\frac{a}{b}$ less than, equal to, or greater than $\frac{c}{d}$ • Comparing quantities: $\frac{a}{b}$ of … less than, equal to, or greater than $\frac{c}{d}$ of … • Using and applying rates: e.g. 4 machines need 17 hours maintenance; how long for 7 machines? ('hours per machine', multiplier $\frac{17}{4}$)	**Phase 3 (three lessons)** Objectives B, C, D, F, G, H **Strategies for solving problems involving multiplication, division, ratio and proportion** Collect a suitable bank of problems, ranging from level 5 to level 7/8. Use Y8 mini-pack problem bank (pp. 23–28), Framework (pp. 3, 5, 21, 25, 75–81, 137, 167, 217, 229, 233, 269), suitable textbooks. Draw from numerical (including percentage increases and decreases), graphical, geometrical (enlargement, maps and plans), statistical contexts. Activities (may include mini-plenaries): • Choose *one* problem: discuss alternative strategies for solving; change numbers (e.g. make them more difficult) and consider how methods can be adapted; ask different or supplementary questions from same context. (See Y8 mini-pack prompts, pp. 19–20.) • Choose a *small set* of problems: concentrate on extracting and organising data (e.g. putting into tabular form) before deciding on possible methods of solution, rather than working problems through to an answer. (See Y8 mini-pack prompts, pp. 21–22.) • Ask pupils to *make up* similar problems for a partner to solve. • Give *part solutions* and ask pupils to continue and complete solution, or give a *complete solution* and ask pupils to evaluate efficiency of strategy chosen and/or identify errors. Problem-solving strategies to be revised and developed: • Translate problem into a form which will help with the solution: e.g. extract appropriate data and put in tabular form. • Estimate the answer: ask 'Will it be bigger or smaller?', 'Will it be greater or less than 1?', etc.; use knowledge of the effect of multiplying or dividing by numbers greater than or less than 1. • Consider scaling methods by finding a multiplier. • When using unitary method involving division, clarify rates expressed part way to a solution, e.g. 'Is it euros per pound or pounds per euro?'	Phase 3 of the Year 8 multiplicative relationships unit outlined a general approach to developing problem-solving strategies. Set the expectations for a Year 9 group by collecting a suitable problem bank. Give particular attention to: • problems involving rates and change of units; • problems involving percentages; • the common structure (relationship of proportionality) applying to different contexts. (See Year 9 phase 3 prompts, pp. 25–27). **Support:** Choose easier examples from the collection. **Extension:** Include problems where the relationships involved are not proportions.	Mini-plenaries and short final plenaries should not introduce new problems, but reflect on: • successful strategies and their transferability; • efficiency of strategies; • emerging misconceptions; • developing coherent written arguments. Use the plenary to assess progress and decide the focus for the next lesson.

Prompts for oral and mental starters: phase 1

The following activities aim to *automate* some of the processing tasks which are essential to speedy solution of ratio and proportion problems. In this sense they are similar to the kinds of activities primary teachers use with Key Stage 2 pupils in order to secure rapid responses to problems involving multiplication and division. This does not exclude *understanding* but seeks to add *speed* to pupils' use of basic knowledge.

It is important to incorporate both pace and discussion within these starters. Generally a few fairly speedy responses should be sought before results are compared. Pupils are then in a position to exchange strategies and improve on responses, ready for another quick batch of questions.

Higher/lower

This is a quick-paced activity requiring a mental approach. Pupils could respond by showing one of two cards – 'higher' or 'lower'.

- Give a scale factor (perhaps point to one on the board).
- Ask whether the scaling will give a result that is **higher or lower** than the original value.
- Part way through, ask pupils to explain how the value of the scale factor helps to indicate the size of the result.
- Introduce the term **reciprocal** and write reciprocals of the values used so far.
- Ask the same set of questions for the reciprocals and note the difference in the direction of the scaling between the reciprocal pairs.
- Suggested scale factors are given here but this list should be varied.
- An alternative is to specify 'higher' or 'lower' and have pupils suggest scale factors using a whiteboard or equivalent.

Scale factor
$\frac{2}{5}$
$\frac{35}{2}$
$\frac{9}{5}$
$\frac{81}{100}$
$\frac{1}{7}$
$\frac{101}{50}$
$\frac{9}{200}$
$3\frac{1}{4}$
$\frac{19}{3}$

About

This is a quick-paced activity requiring a mental approach. Pupils could respond by using whiteboards or equivalent.

- Give a value and a scale factor (perhaps point to these on the board; see the table on the right).
- Initially use only fractions, then only decimals, then a mixture.
- Ask '**About** how big will the resulting value be?'
- Part way through ask some pupils to explain their strategies in general for particular scale factors, e.g. 'How do we find the approximate effect of scaling anything by a factor of $\frac{8}{3}$?'
- Suggestions are given here but this list should be varied.
- Alternatively specify the approximation and ask for the scale factor. Which scale factors would give a result which is:
  - about half of the original?
  - about the same as the original?
  - about twice the size of the original?
  - about ten times the original?

Values to scale	Scale factor	Scale factor
5	$\frac{3}{7}$	0.4
1.6	$\frac{8}{3}$	1.7
29	$\frac{5}{4}$	1.9
240	$\frac{9}{4}$	0.8
0.87	$\frac{19}{3}$	0.53
11.2	$\frac{7}{6}$	1.02
	$\frac{4}{5}$	0.09
	$\frac{4}{13}$	3.3
	$\frac{10}{7}$	9.7

Calculator quick

This activity, which requires the use of calculators, needs a measured pace. Pupils could respond using whiteboards or equivalent. An OHP calculator is useful for the follow-up discussion.

- Give a calculation, perhaps orally or revealed on the OHP. See the suggestions opposite.
- Pupils use their calculators to perform a quick calculation and write their solutions on whiteboards or equivalent.
- Ask a pupil to scribe a selection of the responses on the board.
- Encourage pupils to use arguments of approximation to confirm the correct answer.
- Discuss the key sequence for each calculation (perhaps demonstrated on the OHP calculator). Particular attention could be paid to key sequences which give incorrect solutions if these are common to a number of pupils.
- If the preceding points have not been lengthy, ask what other form the calculation could take in order to prompt the same key sequence.

Calculation
$\frac{25}{4}$
$\frac{3}{7} \times 5$
$6 \div \frac{2}{9}$
$9 \times \frac{6}{7}$
$\frac{12}{3/4}$
$\frac{2}{5} \times \frac{3}{2}$
$\frac{3}{9} \div 5$
$\frac{7}{4} \times 6$
$18 \times \frac{9}{4}$

Prompts for oral and mental starters: phase 2

Equivalent lists

In this quick-paced activity, mental approaches should be used, but calculators may be needed for fraction-to-decimal conversion. Pupils could respond using whiteboards or equivalent.

An OHT with copies of the scaling line segments resource sheet may help in the discussion of why percentage increases and decreases are different.

Start the activity by giving an initial ratio and ask for a full list of equivalent ways of stating this relationship. For example:

The ratio *p* : *q* is	**4 : 3**
What is the scaling from p to q:	
• as a fraction?	$\frac{3}{4}$
• as a decimal?	0.75
Is this an increase or a decrease?	decrease
What percentage decrease does this represent?	25% of p
Inverse	
The ratio q : p is	3 : 4
What is the scaling from q to p:	
• as a fraction?	$\frac{4}{3}$
• as a decimal?	1.33…
Is this an increase or a decrease?	increase
What percentage increase does this represent?	$33\frac{1}{3}$% of q

Webs

In this quick-paced activity, a mental approach or calculator methods could be used, but pupils should be told which is expected. Pupils could respond on a large sheet of paper showing their web or contribute to a class web.

- Give an equivalence (or approximate equivalence) between two values. See the suggestions on the right.
- Ask pupils to generate more equivalences from the given fact.
- Link these equivalences in a web diagram.
- Ask pupils to comment briefly on how particular entries were calculated.
- Include entries for a unit value on each side of the equivalence and highlight these as rates.
- Be explicit in the language to describe rate: '… per …'.
- Draw out the importance of certain key entries in enabling a whole string of other entries to be generated and linked in the web.

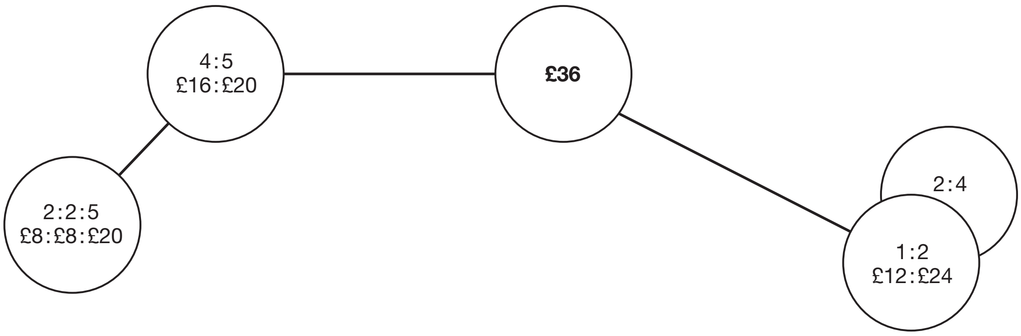
Starting points

- £5 = €8 (or current equivalence)
- 2 gallons = 9 litres
- 5 miles = 8 km
- 5 kg = 11 lb

Splits

Mental approaches should be used in this quick-paced activity. Pupils could respond on a large sheet of paper showing their web or contribute to a class web.

- Give a quantity including units. Examples could include:
  £36, 24 kg, 100 cm, 4.2 ml
- Ask pupils to split the quantity into given ratios.
- Record the splits using a web diagram (see below).
- Ask pupils to comment briefly on how particular entries were calculated.
- Draw out the equivalence of some splits and the links between others. Show these equivalences and links on the web.
- When appropriate draw out a general method for dividing in a given ratio.



Prompts for oral and mental starters: phase 3

Where appropriate:

- expect rapid conversion between fractions (lowest terms), decimal fractions, percentages;
- expect rates, which should be clearly stated as '… per …';
- expect the use of a calculator where numbers are 'awkward'.

Where a proportion is asked for this could be a fraction, a decimal fraction or a percentage.

Expressing proportions

Numbers, first *smaller* than second:

- What is 5 as a fraction of 85?
- What is 5 as a percentage of 85?
- What is 5 as a proportion of 85?

Numbers, first *larger* than second:

- What is 15 as a fraction/percentage/proportion of 8?

Quantities, first *smaller* than second (answer is not in units, e.g. £):

- What is £3 as a fraction/percentage/proportion of £17?

Quantities, first *larger* than second (answer is not in units, e.g. kg):

- What is 23 kg as a fraction/percentage/proportion of 8 kg?

Quantities (mixed units), first *smaller* than second (answer is not in units, e.g. £ or p):

- What is 60p as a fraction/percentage/proportion of £2?

Quantities (mixed units), first *larger* than second (answer is not in units, e.g. m or cm):

- What is 3 m as a fraction/percentage/proportion of 70 cm?

Expressing proportions and their inverses

As above, deal with various forms: fractions/decimal fractions/percentages; numbers/quantities; same units/mixed units.

Examples:

- What is 6 as a proportion of 25?
- What is 25 as a proportion of 6?

Expressing rates

These should be written as 'this per that'. Examples:

- 367 miles travelled at a constant speed for 4 hours
- £87 split equally between 6 children
- 14 hours of work to be covered by 9 clerks

Inverting rates

Using the previous examples:

- What is the meaning of 'that per this'? When is it useful?
  - hours per mile
  - children per pound
  - clerks per hour

Finding proportions of …

Numbers, proportions *less* than 1, some terminating decimals and some not:

- What is three fifths of 83?
- What is 0.6 of 83?
- What is 60% of 83?

Numbers, proportions *greater* than 1, some terminating decimals and some not:

- What is ten sixths of 74?
- What is 1.67 of 74?
- What is 167% of 74?

Quantities, proportions *less* than 1, some terminating decimals and some not (answer is in units, e.g. £):

- What is three sevenths of £17?

Quantities, proportions *greater* than 1, some terminating decimals and some not (answer is in units, e.g. litres):

- What is four thirds of 14 litres?

Comparing proportions

Numbers, first *smaller* than second:

- What is greater:
  5 as a proportion of 85, or
  7 as a proportion of 90?

Quantities, first *smaller* than second (answer is not in units, e.g. minutes):

- What is greater:
  5 minutes as a proportion of 18 minutes, or
  3 minutes as a proportion of 10 minutes?

Quantities (mixed units), first *smaller* than second (answer is not in units, e.g. £ or p):

- What is greater:
  60p as a proportion of £2, or
  90p as a proportion of £2.47?

Numbers, first *larger* than second:

- What is greater:
  26 as a proportion of 8, or
  32 as a proportion of 9?

Quantities, first *larger* than second (answer is not in units, e.g. cm):

- What is greater:
  18 cm as a proportion of 7 cm, or
  16 cm as a proportion of 6 cm?

Quantities (mixed units), first *larger* than second (answer is not in units, e.g. m or cm):

- What is greater:
  3 m as a proportion of 70 cm, or
  2.3 m as a proportion of 57 cm?

Inverting proportions and comparing

As above, deal with various forms: fractions/decimal fractions/percentages; numbers/quantities; same units/mixed units.

Example:

- What is greater:
  9 as a proportion of 28, or
  13 as a proportion of 40?

  Given the previous fact, what is greater:
  28 as a proportion of 9, or
  40 as a proportion of 13?

Comparing rates

- What is the greater rate:
  367 miles travelled at a constant speed for 4 hours, or
  640 miles travelled at a constant speed for 7 hours?

Comparing quantities

Numbers, proportions *less* than 1, some terminating decimals and some not:

- Which is the greater:
  one fifth of 83 or
  one seventh of 90?

Numbers, proportions *greater* than 1, some terminating decimals and some not:

- Which is the greater:
  five thirds of 35 or
  seven quarters of 33?

Quantities, proportions *less* than 1, some terminating decimals and some not (answer is in units, e.g. litres):

- Which is the greater:
  16% of £25 or
  25% of £16?

Quantities, proportions *greater* than 1, some terminating decimals and some not (answer is in units, e.g. litres):

- Which is the greater:
  ten thirds of 14 litres or
  ten sevenths of 20 litres?

Using and applying rates

- 4 machines need 17 hours of maintenance. How many machines can be serviced in 5 hours?

  What is the rate we use in this calculation? (Not expecting numerical answer but a rate, e.g. 'machines per hour')

  What is the calculation? (Not expecting numerical answer but a calculation, e.g. $\frac{4}{17} \times 5$)

  What is the answer?

- 4 machines need 17 hours of maintenance. How many hours are needed to service 7 machines?

  What is the rate we use in this calculation? (Not expecting a numerical answer but a rate, e.g. 'hours per machine')

  What is the calculation? (Not expecting numerical answer but a calculation, e.g. $\frac{17}{4} \times 7$)

  What is the answer?

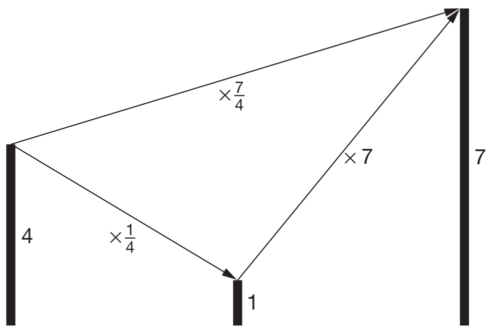
Prompts for main activities in phase 1

Repeated scaling

In the Year 8 unit, pupils used the image of parallel number lines to explore multiplicative relationships. This image is continued, but simplified to the use of line segments. Note that:

- The diagrams can be rough sketches.
- It helps to think in terms of 'expansion' and 'reduction' of the line segment.
- Depending on the class, you might introduce the mathematical term 'enlargement', for example:
  - expansion by a factor of 4 is equivalent to enlargement by a scale factor of $\times 4$;
  - reduction by a factor of 4 is equivalent to enlargement by a scale factor of $\times \frac{1}{4}$.

This will help pupils to make a link with enlargement in phase 2 of the unit. It will of course be necessary to discuss the apparent contradiction in using the term 'enlargement' for a reduction!

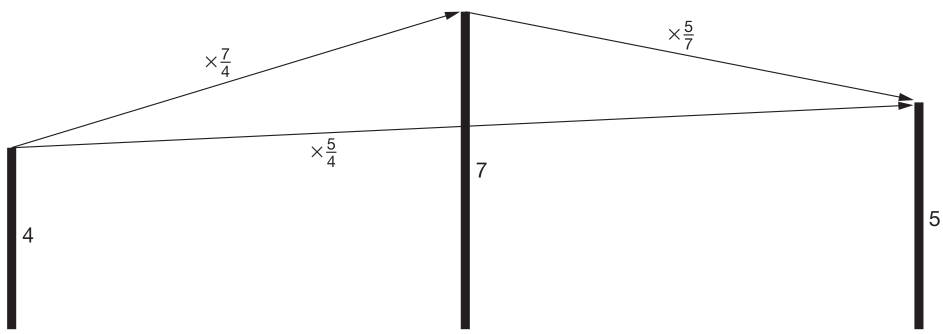


Quickly recap the principle of scaling between any two numbers in either direction using only multiplication. For most pupils the interim steps of scaling down to 1 will be dropped after the initial recap.

For example, ask:

- What single number could you multiply 4 by to scale it to 1?
- What about the inverse… from 1 to 4?
- What single number could you multiply 1 by to scale it to 7?
- What about the inverse… from 7 to 1?
- What number could you multiply 4 by to scale it to 7 in a single step?
- What is the single multiplier coming back from 7 to 4?

Move on to consider successive scalings.



Use the prompts below for a variety of start, middle and end points including non-integer values. For example, ask:

- What single number could you multiply 4 by to scale it to 7?
- What single number could you multiply 7 by to scale it to 5?
- What number could you multiply 4 by to scale it to 5 in a single step?
- How is this related to the two steps we have described from 4 to 7 and from 7 to 5?
- Can you see that multiplying by the single scale factor ($\frac{5}{4}$) is equivalent to multiplying in two steps by the two separate scale factors ($\frac{7}{4}$ and $\frac{5}{7}$)?

$$4 \longrightarrow \boxed{\times\tfrac{7}{4}} \longrightarrow 7 \longrightarrow \boxed{\times\tfrac{5}{7}} \longrightarrow 5$$

$$4 \longrightarrow \boxed{\times\tfrac{7}{4} \times\tfrac{5}{7}} \longrightarrow 5$$

$$4 \longrightarrow \boxed{\times\tfrac{5}{4}} \longrightarrow 5$$

- Use your calculator to check each step of the scaling and to confirm this equivalence. (Point out the rounding error which can occur here.)

$$4 \longrightarrow \boxed{\times 1.75} \longrightarrow 7 \longrightarrow \boxed{\times 0.714} \longrightarrow 5$$

$$4 \longrightarrow \boxed{\times 1.75 \times 0.714} \longrightarrow 5$$

$$4 \longrightarrow \boxed{\times 1.25} \longrightarrow 5$$

- Can you see that the single multiplier (1.25) is the product of the multipliers used in the two-step scaling (1.75×0.714)?
- Repeat the above for the inverse scalings, from 5 to 7 to 4 noting that beginning with the fractional form of the scale factors is useful here.

Link to multiplication and division of fractions

Although not addressed in this unit, these points are worth noting:

- Successive scalings $\times\frac{b}{a}$ and $\times\frac{c}{b}$ can be thought of as expanding by a factor of *b* and reducing by a factor of *a*, then expanding by a factor of *c* and reducing by a factor of *b*. Expansion and reduction by factors of *b* can be observed to ‘undo’ one another. This may help to explain the result $\frac{b}{a} \times \frac{c}{b} = \frac{c}{a}$ in terms more meaningful than ‘cancelling’ the *b*s.
- The inverse of $\times\frac{b}{a}$ is $\div\frac{b}{a}$. However, the inverse can also be expressed as the multiplier $\times\frac{a}{b}$. So $\div\frac{b}{a}$ and $\times\frac{a}{b}$ are equivalent operators, which explains the infamous (familiar but frequently not understood) rule for dividing by a fraction: ‘turn the fraction upside down and multiply’.

Proportional sets

Tables 1 and 2 show proportional sets of numbers. The main activity is to give pupils some of the entries and ask them to calculate the others. A suggestion is given in the incomplete table on the right – clearly enough entries are positioned to make the task possible but also challenging.

Table 1

Set A	Set B	Set C
2	7	3
5	17.5	7.5
8	28	12
0.4	1.4	0.6
0.2	0.7	0.3
$\frac{1}{2}$	$\frac{7}{4}$	$\frac{3}{4}$
10	35	15

Set A	Set B	Set C
2	7	
5		7.5
	28	
		0.6
0.2		
	$\frac{7}{4}$	
10		

Table 2

Set A	Set B	Set C
5	4	11
6.25	5	13.75
3	2.4	6.6
7	5.6	15.4
0.5	0.4	1.1
8	6.4	17.6
10.5	8.4	23.1

Set A	Set B	Set C
5		11
	5	
3	2.4	
7		
		1.1
		17.6
10.5		

The presentation of this task is important. This is not a routine completion of entire rows or columns. The aim is to use the table as an opportunity to draw out the structures which help to develop proportional understanding.

You could model the following approach.

Finding a missing entry

- To find a missing entry, three related entries are required. A good strategy is to set out the entries in a 2-by-2 array and then to identify the multiplier. (The multiplier may be between any pair of numbers.)

$$\times\frac{2}{5}$$

2	x
5	7.5

- Encourage pupils to estimate the size of the missing entry. (Refer back to the oral and mental starters 'Higher/lower' and 'About'.)
- Write the calculation as an equation which is then solved mentally or using a calculator:

  $x = 7.5 \times \frac{2}{5}$

Drawing together solutions and comparing strategies

- Consider alternative arrays and calculations and discuss their merits.
- At a suitable stage consider the scale factor between the columns and its inverse. This can be seen as a 'safety net' by which all missing entries can be calculated if a simpler strategy is not apparent. For example:

  Scaling from A to B involves multiplying by ratio $\frac{b}{a} = \frac{7}{2}$

  (Note also the inverse ratio $\frac{a}{b} = \frac{2}{7}$)

  Scaling from B to C involves multiplying by ratio $\frac{c}{b} = \frac{3}{7}$

  (Note also the inverse ratio $\frac{b}{c} = \frac{7}{3}$)

  Scaling from A to C involves multiplying by ratio $\frac{c}{a} = \frac{3}{2}$

  (The inverse is $\frac{a}{c} = \frac{2}{3}$)

  This is equivalent to the product of the two scalings $\frac{b}{a} \times \frac{c}{b} = \frac{7}{2} \times \frac{3}{7} = \frac{3}{2}$

- Pupils will sometimes use division to complete the calculation, for example:

4	11
?	1.1

÷ 10

  Remind pupils that dividing by 10 can also be thought of as multiplying by $\frac{1}{10}$ or 0.1.

Extending the task by posing further questions

For example:

- Which entries must be placed in the table in order to be able to complete all missing entries?
- What is the maximum number of items which could be entered and the task remain impossible?
- What is the minimum number of entries needed and what is important about their location?
- For any one empty cell – What is the best starting point? How many different starting points are there?

Simplifying the task

- Pre-select the array to be used to find the missing entry.
- Use the scale factor between columns systematically.

Prompts for main activities in phase 2

Folding paper

Two paper-folding activities follow. The first, based on a square, can be used as a practical demonstration or a visualisation exercise. Notes for both are given below. The second, based on a rectangle, is a practical activity.

Folding a square sheet

This is a practical demonstration for the majority of pupils.

Resources: Two identical large square sheets of paper.
A smaller square sheet to check for similarity (not produced by folding the larger ones) and a rectangle (sides in a ratio of about 5 : 1).

Start by showing pupils the identical square sheets. Stick one sheet on the board and fold the other sheet to make a rectangle and cut into two halves. Stick one half on the board and repeat with the other half (folding parallel to a pair of sides, along the shorter line of symmetry). Do this three more times.

Ask volunteers to arrange the six shapes on the board in different ways, e.g. align to show centre of enlargement. Below are some useful points to draw out:

- There are three squares, with sides in the ratio 1 : 1, each successive one being a $\times \frac{1}{2}$ enlargement of the previous square.
- There are three rectangles, with sides in the ratio 2 : 1, each successive one being a $\times \frac{1}{2}$ enlargement of the previous rectangle.
- Shapes whose proportions are the same, one being an enlargement of the other, are described as mathematically *similar*.
- Show pupils the smaller square sheet and ask whether it belongs to the set of similar shapes. Check by measuring (to check ratio) and by aligning (to check centre of enlargement). Conclude that all squares are similar.
- Show pupils the other rectangle (e.g. ratio 5 : 1) and ask whether it belongs to the set of similar shapes. Check by measuring (to check ratio) and by aligning (to check centre of enlargement). Conclude that only certain rectangles are similar and that this depends on the ratio of length to width.

Folding a square sheet: visualisation

This alternative visualisation exercise is suitable for higher-attaining pupils.

For general guidance on conducting visualisation exercises, see the ‘Prompts for oral and mental starters in phase 1’ in the Year 9 geometrical reasoning unit.

Visualisation script

Imagine a large square sheet of paper. Ask yourself: What is the ratio of length to width? (Check that pupils understand that it is 1 : 1.)

Imagine folding it in half along a line of symmetry. What shape do you get? (Take answers from the group – two possibilities are a rectangle and a right-angled isosceles triangle. Ask the class to focus on folding parallel to a side to get a rectangle. *There should be no further discussion until the end of the visualisation.*)

Imagine the rectangle you get after folding a square sheet in half. Ask yourself: What is the ratio of length to width? (Extension: How does the area compare with the area of the square?)

Now fold the rectangle in half along the shorter line of symmetry. What shape do you get? What is the ratio of length to width? (Extension: How does the area compare with the area of the original square?)

Fold again along a line of symmetry (not a diagonal). What shape do you get? What is the ratio of length to width?

Fold again along the shorter line of symmetry. What shape do you get? What is the ratio of length to width?

Fold one more time along a line of symmetry (not a diagonal). What shape do you get? What is the ratio of length to width?

At the end of the visualisation, ask pupils to describe the sequence of six shapes they saw and the relationships between them. You may wish to have a set of shapes prepared to demonstrate and check pupils' observations. Draw out the same points as for the practical activity.

Folding a rectangular sheet

Ask pupils to work in pairs and split the class into two halves. Provide one half of the class with two sheets of A4 paper and the other with two identical sheets of non-metric paper (e.g. trimmed A3). Ask them to set one sheet aside and to fold the other sheet along the shorter line of symmetry and cut it into two halves. Set one half aside and repeat with the other half. Do this three more times.

Now ask pupils to:

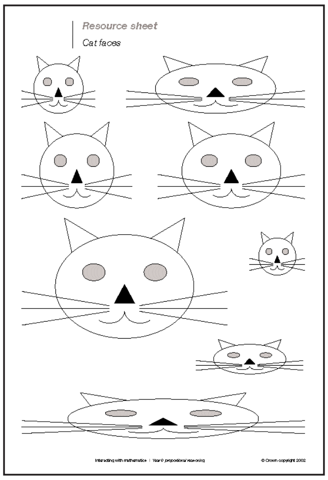
- Arrange the sheets in different ways and consider whether any of the rectangles are mathematically similar (e.g. align to show centre of enlargement).
- Measure and tabulate lengths and widths of rectangles which they think are similar and calculate scale factors within the table.

On the board compare the results from the two halves of the class, making sure that you display accurate data. Allow all pairs a few minutes to consider the reasons for the difference (they may wish to form fours to discuss the two sets of results).

In the plenary raise these questions for discussion:

- Considering the non-metric sheet, are any rectangles mathematically similar?
- What happens with A4 paper?
- Why do you think that standard metric paper sizes are designed in this way? (You have been looking at A4, A5 and A6, and smaller sizes which are not named. They are derived from A0 paper, which has an area of 1 m^2.)

Cat faces

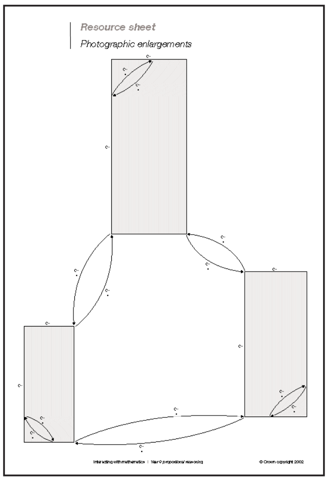


Distribute copies of the ‘Cat faces’ resource sheet to pairs of pupils. If you use a selection of sizes of sheet (e.g. A3, A4, A5), additional questions on the preservation of ratios across the enlarged sheets will be possible when pupils are ready to share their results.

Describe the following tasks:

- Find the faces which are similar by calculating the ratio of two dimensions within a face and confirming that this is preserved.
- Record the factor which scales from one similar face to another (as a fraction, decimal and percentage).
- Record the inverse of each of these.
- Check some additional measurements on the similar faces and confirm that they also have increased by the scaling identified.
- Consider the scalings between the three circular faces A, B and C. Confirm that the scaling from A to C can be calculated by multiplying the scaling from A to B and the scaling from B to C.

Photographic enlargements



Show the set of similar rectangles given on the resource sheet ‘Photographic enlargements’ (OHT, poster size or A3 sheets).

Explain that:

- This is a set of photographs which are enlargements of each other (i.e. it is a set of similar rectangles).
- There are 18 values missing from the diagram (on a poster you might like to have these values hidden by small sticky notes – ‘lift the flap’), but they cannot be found by measuring as the rectangles are not drawn accurately.

- Of the missing values:
  - six are dimensions – the width and length of each rectangle;
  - six are scalings between the three rectangles in both directions;
  - six are internal ratios written as scalings from the length to the width and vice versa.

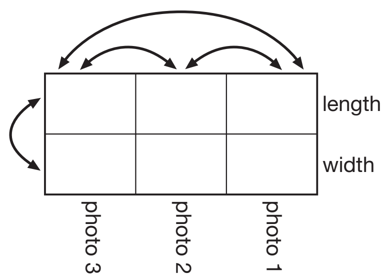
Pose the challenge:

> I will give you some of the values. You have to ask for the *minimum* you need to work out all the other values. What values do you want me to give? (Which flaps would you choose to lift?)

The difficulty of this task is largely determined by the decision making required as to which numbers are revealed. More guidance on this choice could be given to support some groups of pupils.

Follow up:

- Pupils should explore alternative sets of given values and consider what is the least number they need.
- They might tabulate dimensions and the scalings between them as follows:



- Extension: What is the largest number of values you could be given and still be unable to find all 18 values?

Prompts for main activities in phase 3

Phase 3 of the Year 8 multiplicative relationships unit outlined a general approach to developing problem-solving strategies. The development of these strategies is continued here. You need to collate a bank of appropriately pitched Year 9 problems. Important sources for this will be questions from past Key Stage 3 test papers, textbooks and the Framework supplement (pages 3, 5, 21, 25, 75–81, 137, 167, 217, 229, 233, 269). The problem bank included in the Year 8 materials (pages 23–28 in the multiplicative relationships mini-pack) is still likely to be a useful resource for this unit.

Use the ideas below for further guidance appropriate to the Year 9 context. Also refer to the Year 8 'Prompts for main activities in phase 3' on pages 19–22 of the multiplicative relationships mini-pack.

Rates and change of units

Example 1: Speed

A car averages 56 mph. How far does it travel in $3\frac{1}{4}$ hrs? To the nearest $\frac{1}{4}$ hr, how long does it take to travel 322 miles?

This example is typical of many problems where a rate is given. Many pupils will approach these problems successfully without tabulation recognising that a multiplication or division is required. However, a discussion of the question in its tabular form can be of value to all pupils in clarifying the relationships in the problem.

- Extract the data into a tabular form, noting that one of the entries will be 1.
- Support by simple questions such as those rehearsed in the oral and mental starters.

Distance (miles)	Time (hours)
56	1
?	3.25
322	?

*The first answer is a distance, so the **units** will be miles.*
*It will be **bigger** than 50; it will be **about** 150 miles.*
*The second answer is a time. The **units** will be hours.*
*The answer will be **bigger** than 3.25.*
*It will be **about** 6.*

Example 2: Density

The density of paint is 1.3 g/cm^3. What is the mass of 3 litres of paint?

In this example, there is a change of units. To highlight the structure of the question the data can again be tabulated. This may initially be solved as a two-stage calculation. It might be appropriate to combine the two scale factors into one if the calculation is to be repeated for a series of quantities or for tabulation in a spreadsheet.

Mass (g)	Volume (cm^3)	Volume (litres)
1.3	1	
	1000	1
?	?	3
$\xleftarrow{\times 1.3}$	$\xleftarrow{\times 1000}$	

$\xleftarrow{\times 1300}$

*The first row shows a **rate** given in the question. This will change for different paint.*
*The second row shows a conversion **rate**. This will always be the same.*
*The final answer is a mass. The **unit** will be grams.*
*The answer will be **bigger** than 1.3; it will be **between** 3000 and 4500 grams.*

Percentage changes

Example 3: Savings

a What is the final amount after 7% interest is added to my savings of £528?

b My friend's savings amount to £920 after 7% interest has been added. What was the original amount of her savings before interest was added?

Terms like 'increase' (or 'decrease') may trigger additive thinking, but it is always possible and sometimes necessary to relate the original and final values multiplicatively.

Part a:

Original	Final
100	107
528	?
?	920

This table gives a concise overview of the problem. The first row shows which amount to consider as 100%. All '?' entries are amounts of money; the ***units*** *will be £. Each entry in the first column will be* ***smaller*** *than the corresponding entry in in the second column.*

$$528 \xrightarrow[\times 1.07]{\times \frac{107}{100}} \text{final amount}$$

$$\text{final amount} = 528 \times \frac{107}{100} = 528 \times 1.07$$

From any original amount (£s) the final amount (£t) can be calculated:

$$t = s \times 1.07$$

Part b: The second part of this inverse calculation is made possible because of the approach through considering scaling methods and finding a multiplier. Discuss the fact that the reduction here is not 7% but about 6.5%:

$$\text{original amount} \xleftarrow[\times 0.935]{\times \frac{100}{107}} 920$$

$$\text{original amount} = 920 \times \frac{100}{107} = 920 \times 0.935$$

From any final amount (£t) the original amount (£s) can be calculated:

$$s = t \times 0.935$$

Ask pupils to make up similar problems for their neighbour to solve – using different final amounts and different interest rates.

Example 4: Prices

a If an item originally selling for £30 is reduced by 50% and then by a further 50%, do I get it for nothing?

b If the price is increased by 50% and then decreased by 50%, does it return to its original value?

Misconceptions relating to successive percentage changes can be exposed here if pupils answer 'yes' to either of these questions. This usually results from additive rather than multiplicative thinking. The tables opposite give a concise overview of the problem. The first two rows show which **amount to consider as 100%.**

Part a

First	Second	Third
100	50	
	100	50
30	?	?

*Note that a **single scaling** of 0.5 × 0.5 = 0.25 reveals that the third price is 25% of the first price.*

third price = £30 × 0.5 × 0.5 = £7.50

Part b

First	Second	Third
100	150	
	100	50
30	?	?

*Note that a **single scaling** of 1.5 × 0.5 = 0.75 reveals that the third price is 75% of the first price.*

third price = £30 × 1.5 × 0.5 = £22.50

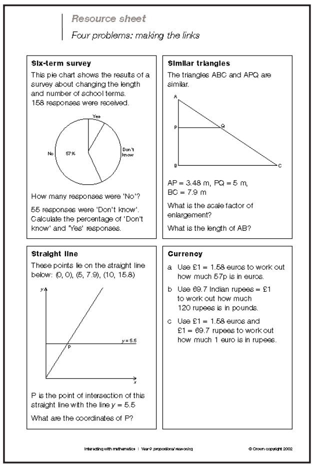
Links between different strands of mathematics

List the aspects of mathematics with which pupils are familiar (it is probably better not to include 'using and applying mathematics' at this point):

- number
- algebra
- shape, space and measures
- handling data

For each aspect ask the class:

- What is it about?
- What sorts of problems do you solve in this aspect of maths?
- What types of table, diagram or graph are used?



Resource sheet

Four problems: making the links

Six-term survey

This pie chart shows the results of a survey about changing the length and number of school terms.
158 responses were received.

How many responses were 'No'?

55 responses were 'Don't know'. Calculate the percentage of 'Don't know' and 'Yes' responses.

Similar triangles

The triangles ABC and APQ are similar.

AP = 3.48 m, PQ = 5 m, BC = 7.9 m

What is the scale factor of enlargement?

What is the length of AB?

Straight line

These points lie on the straight line below: (0, 0), (5, 7.9), (10, 15.8)

P is the point of intersection of this straight line with the line $y = 5.5$

What are the coordinates of P?

Currency

a Use £1 = 1.58 euros to work out how much 57p is in euros.

b Use 69.7 Indian rupees = £1 to work out how much 120 rupees is in pounds.

c Use £1 = 1.58 euros and £1 = 69.7 rupees to work out how much 1 euro is in rupees.

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Use the resource sheet 'Four problems: making the links' as a handout, OHT or poster.

For each problem ask pupils to:

- discuss from which strand of mathematics the question could be drawn;
- extract the data and put it into tabular form – do not solve the problem at this stage;
- discuss ways of constructing a table (with headings);
- compare any alternatives and decide which is best.

Considering the collection of tables for all four problems, ask pupils to:

- compare tables for different problems, noting the similarity in structure, each being a proportion in which the answer can be found by finding and applying a scale factor;
- continue the problems through to solutions.

Finally point out that these examples illustrate how an important mathematical idea (proportion) can underpin apparently very different situations.

Prompts for final plenary in phase 2

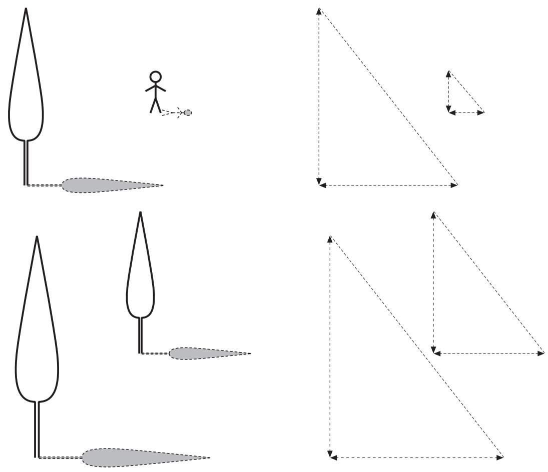
Shadows

This is suggested as an extended plenary to conclude phase 2 of the unit.

Resources: The four resource sheets 'Shadows' will need to be copied on to acetate. You will find these sheets in the school file.

Show 'Shadows 1' which is a sketch of a child, some trees and their shadows. Ask pupils what they notice about the figures and their shadows and how they can tell that the shadows are all cast at the same time of day. Discuss the general question of using the height of the child and length of the child's shadow to establish something about the relationship between the height of each tree and the length of its shadow.

Use 'Shadows 1 overlay' to illustrate the measurements which are known (lengths of the shadows and height of the child) and those which need to be calculated (heights of the trees). For simplicity 'Shadows 1' can be removed and the overlay left in place to minimise distractions in the display.



Use the task as an opportunity to reinforce the strategies which have been developed throughout the unit, namely:

- Write the three known values and the one unknown as a 2-by-2 array and decide which scaling will help find the unknown (using ideas from phase 1).
- Use the multiplier to decide whether the calculation will result in a higher or lower result and also establish about how big the result will be (using ideas from the oral and mental starters).
- Confirm this by reference to the context of the question.
- Use calculators to quickly find the height of each tree and reinforce efficient calculator use (ideas from oral and mental starters).

Show 'Shadows 2', a sketch of the same trees and the same child, and ask what is noticeable about the shadows now. Are any of the relationships from the first situation transferable to this new context? Ask how much information is needed here to calculate the lengths of the shadows.

Use 'Shadows 2 overlay' and the key points described above to calculate the length of the shadow of each tree. Note that part of the picture has deliberately 'fallen off' the sheet. At this stage, relationships are well enough established that it is still possible to find a solution by visualising the complete triangle.

