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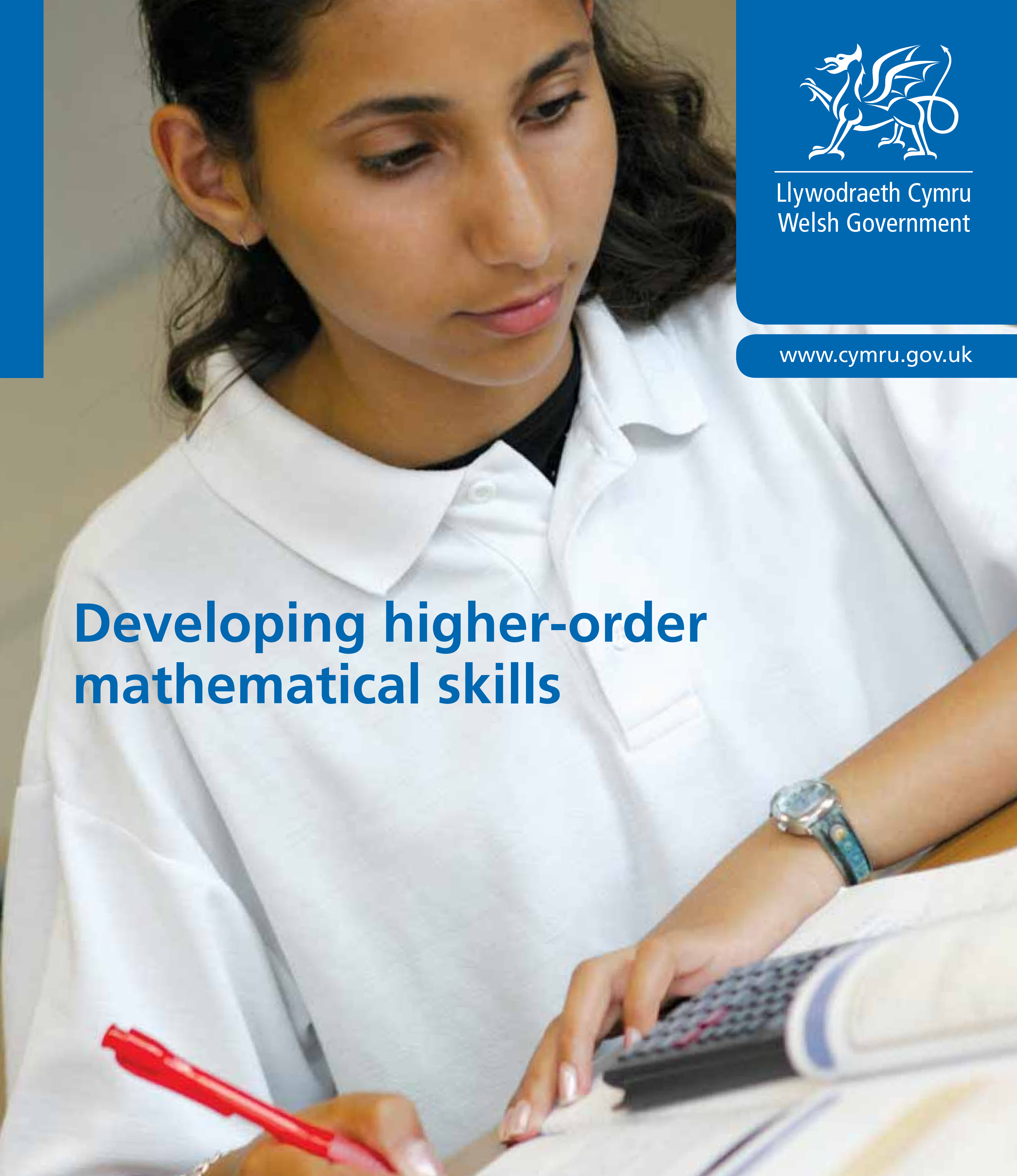
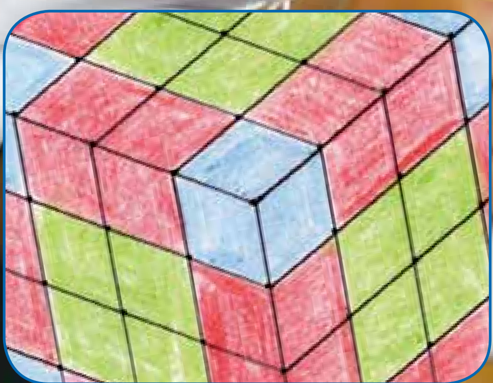
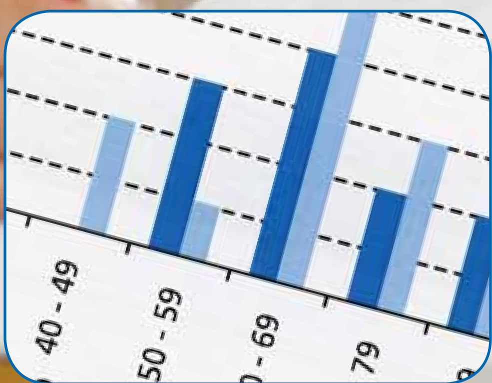


Developing higher-order mathematical skills





Developing higher-order mathematical skills

Audience	Secondary school mathematics teachers and senior managers; local authorities; national bodies with an interest in education.
Overview	This document is designed to assist teachers to recognise and promote higher-order mathematical skills within Key Stage 3 and through to Key Stage 4. It provides examples of learners' work showing characteristics of Level 7 to Exceptional Performance (EP) within the national curriculum for mathematics. The examples are accompanied by commentary that identifies the characteristics of higher-order mathematical skills.
Action required	Schools' senior managers and subject leaders, and local authority advisers, are requested to raise awareness of this new resource within their mathematics departments, and to encourage teachers to use the materials to support their focus on securing and improving learners' mathematical skills.
Further information	Enquiries about this guidance should be directed to: Curriculum Division Department for Education and Skills Welsh Government Cathays Park Cardiff CF10 3NQ Tel: 029 2082 1750 e-mail: curriculumdivision@wales.gsi.gov.uk
Additional copies	Can be obtained from: Tel: 0845 603 1108 (English medium) 0870 242 3206 (Welsh medium) Fax: 01767 375920 e-mail: DfESWales1@prolog.co.uk Or by visiting the Welsh Government's website www.wales.gov.uk/educationandskills
Related documents	*Mathematics in the National Curriculum for Wales; Making the most of learning: Implementing the revised curriculum (Welsh Assembly Government, 2008); Mathematics: Guidance for Key Stages 2 and 3* (Welsh Assembly Government, 2009) This document is also available in Welsh.

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Contents

Foreword

Mathematical skills are fundamental to success in life for everyone. They underpin effective learning in all subject areas across the curriculum. Mathematical skills are vital to further progress in a wide range of disciplines and unlock the doors to employment, helping people to become active citizens in today's society.

Results of national and international assessments – Key Stage 3, GCSE and PISA in particular – point to the fact that some more able students in Wales are not generally developing their mathematical skills to their full potential.

This guidance document has been created in collaboration with a number of recognised experts in schools and local authorities across Wales. The guidance is intended to support mathematics teachers working with able learners in Key Stages 3 and 4, to help their learners achieve better results at the end of Year 9, at GCSE and beyond.

At a national level we have done a great deal over the years, through a range of initiatives, to support learners who have the least well-developed mathematical skills. So far, we have not given as much attention to boosting the mathematical skills of our higher-attaining learners.

This publication is aimed at filling that gap and I am delighted to commend it to all schools and local authorities, and to confirm our support for higher-order skills as part of the wider focus on effective schools.

Chris Tweedale
Director of Schools and Young People Group
Department for Education and Skills

Introduction

Why has this guidance been produced?

The Welsh Government is committed to challenging underachievement in schools. This new guidance focuses on raising the performance of mathematical skills for all learners during Key Stage 3, and into Key Stage 4. In particular, it supports mathematics teachers to meet the needs of their most able learners.

Underachievement occurs in learners of all abilities, although it is perhaps most easily identified in the work of the less able. A considerable number of strategies are employed in schools to raise the standard of work of learners who are underachieving, often through differentiated work that targets learners working at Levels 3 and 4 in a mixed-ability class or through teaching these learners in small groups with specialised support. It is less common for teachers to target more able learners who might also be underachieving, even though their attainment is at the expected level or above. If we are to raise performance, it will be necessary to raise expectations by targeting those who are 'coasting' and challenging them to show their true potential.

The examples of work in this guidance aim to exemplify what Key Stage 3 learners, working at the highest levels in mathematics, can achieve. The examples provide commentaries that will help teachers to identify characteristics of Levels 7, 8 and Exceptional Performance. They are intended to provide a stimulus for learning and teaching, and present suggestions for transition to related post-14 work.

The Department for Education and Skills (DfES) curriculum guidance, *Mathematics: Guidance for Key Stages 2 and 3* (Welsh Assembly Government, 2009), provides key messages for planning learning and teaching in mathematics to support Curriculum 2008. It includes learner profiles exemplifying how to use level descriptions to make best-fit judgements at the end of Key Stages 2 and 3.

Many of the key messages from the curriculum guidance are further developed in this higher-order mathematics booklet, and are therefore relevant to **all** learners. However, there are obvious differences, in that this new guidance provides commentaries on recognising characteristics of level descriptions in individual examples of learners' work, rather than focusing on end of key stage judgements.

In Section 1 of this booklet is an explanation of the changes to the GCSE mathematics examinations to be first awarded in summer 2012, to reflect the revised Key Stage 4 subject Orders. Some exemplar questions from recent WJEC pilot GCSE examinations are also included. Also in this booklet is a section on the implications for Wales of the disappointing results from the Programme for International Student Assessment (PISA). The nature of many of the questions used in these assessments is considered and some sample questions are included. Finally, this section includes a description of the implications of these assessments for teachers in challenging their learners to reach their potential.

Using this guidance

This booklet is divided into two sections.

Section 1	describes the latest developments in GCSE examinations, reflects on the performance of Welsh learners in the PISA tests and describes the implications for teachers in ensuring their learners are able to meet the demands of these assessments in future.
Section 2	contains examples of learners' work exemplifying higher-order level characteristics. The contexts and skills used by learners are then linked with possibilities for further development in Key Stages 3 and 4.

This guidance is for mathematics teachers to:

- extend their understanding of the mathematics Order 2008
- review learning plans and activities
- consider the characteristics of level descriptions set out in the mathematics Order 2008
- work with other teachers to reach a shared understanding of the level descriptions
- develop departmental portfolios to exemplify characteristics of level descriptions
- develop departmental learner profiles to exemplify end of key stage best-fit judgements
- prepare learners to cope with revisions to GCSE examinations.

This guidance is part of a range of materials that will help teachers to implement the revised curriculum and its associated assessment arrangements. This includes materials focused on mathematics and also on the wider aspects of effective learning and development of skills. Pages 81–83 provide a list of useful references for mathematics teachers.

Section 1

GCSE and PISA, and the implications of these assessments

GCSE and PISA, and the implications of these assessments

Changes in GCSE examinations from 2011/12

Mathematics in the National Curriculum for Wales, the 2008 mathematics Order, places an increased emphasis on the skills that were formerly referred to as 'Using and Applying Mathematics'. The GCSE specifications in mathematics have been revised to reflect this change of emphasis. These skills will be assessed through written examination questions, and revised GCSEs will be first awarded in summer 2012. Teachers in Wales will need to prepare their learners for these new assessments.

The aims and learning outcomes of the revised GCSE specifications will enable learners to:

- develop skills, knowledge and understanding of mathematical methods and concepts
- acquire and use problem-solving strategies
- select and apply mathematical techniques and methods in mathematical, everyday and real-world situations
- reason mathematically, make deductions and inferences, and draw conclusions
- interpret and communicate mathematical information in a variety of forms appropriate to the information and context.

The revised Assessment Objectives (AOs) are shown in the table below, together with their respective weightings across the qualification.

Assessment Objectives		Weighting (%)
AO1	Recall and use their knowledge of the prescribed content	45–55
AO2	Select and apply mathematical methods in a range of contexts	25–35
AO3	Interpret and analyse problems and generate strategies to solve them	15–25

As can be seen from this table, the proportion of marks allocated to Assessment Objectives 2 and 3 is approximately 50 per cent. In order to reflect this weighting, GCSE examination papers will contain an increased proportion of contextualised questions and questions that require the use of problem-solving strategies for their solution. Teachers will need to ensure that their teaching prepares their learners to be able to tackle such questions. The removal of the coursework component from GCSE mathematics released more lesson time for the teaching of mathematics. This should have enabled teachers to find the time to enhance their teaching by adopting a problem-solving approach, which in turn would prepare their learners for the new styles of examination question.

Some questions will be set in more complex real-life contexts than has been the case in recent years. These will require more explanatory text and will place a greater demand upon candidates' skills of reading and comprehension. A greater proportion of questions will be unstructured, providing all the necessary information and leaving the candidates to find their own ways through. For these questions, there will often be more than one method of solution, and candidates will be expected to devise their own strategies. Without a problem-solving approach to teaching and learning, candidates are likely to find difficulty in engaging with these questions.

Some exemplar questions, taken from WJEC pilot GCSE examination papers, together with their mark schemes, are shown on the following pages.

Exemplar pilot GCSE questions

WJEC Pilot GCSE Mathematics, Summer 2009, Higher Tier Paper 2, Question 13

A rectangular shape is made using 12 square tiles placed with equal gaps between them. The overall length of the rectangle is 645 mm and the overall width is 475 mm.

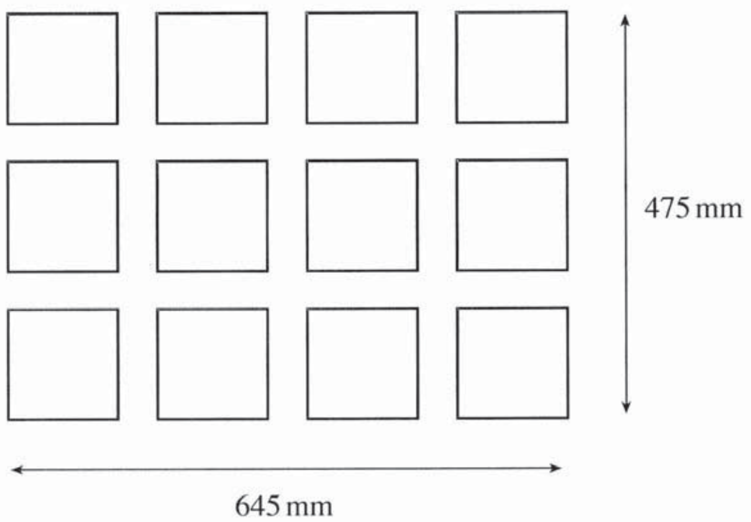


Diagram not drawn to scale.

Find the dimensions of each tile and the width of each gap in mm.

Mark scheme

Summer 2009 Paper 2 Wales Pilot Higher Tier	Mark	Comments
$3x+2y = 475$ OR $645 - 475 = 1$ tile + 1 gap $4x + 3y = 645$ OR Tile + Gap = 170	B1 B1	Or other strategy *Pairs of values may imply these first B marks*
(Tile) 135 (mm) AND (Gap) 35(mm)	B2	CAO. B1 for either correct value

WJEC Pilot GCSE Mathematics, Summer 2009, Higher Tier Paper 2, Question 14

A rectangle is shown in the diagram between two parallel lines.

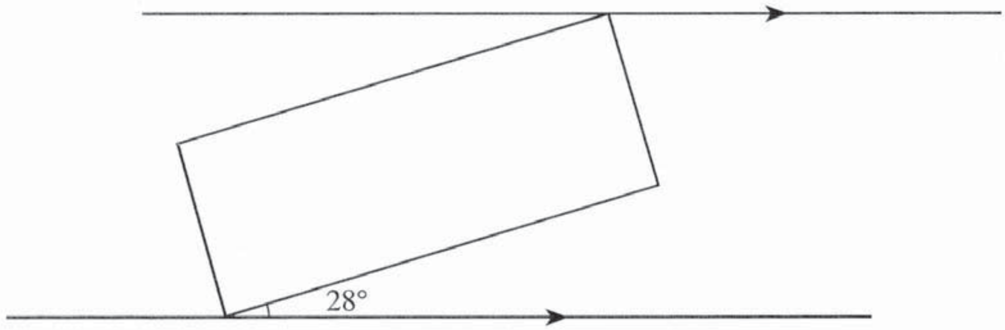


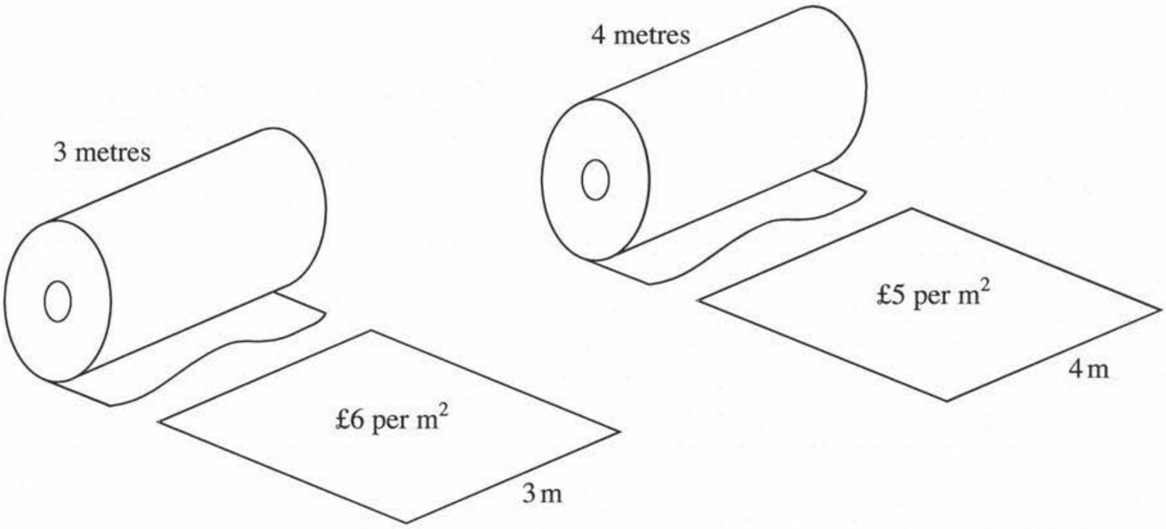
Diagram not drawn to scale.

The rectangle is of length 7·1 cm and width 3·4 cm.
Calculate the perpendicular distance between the parallel lines.

Mark scheme

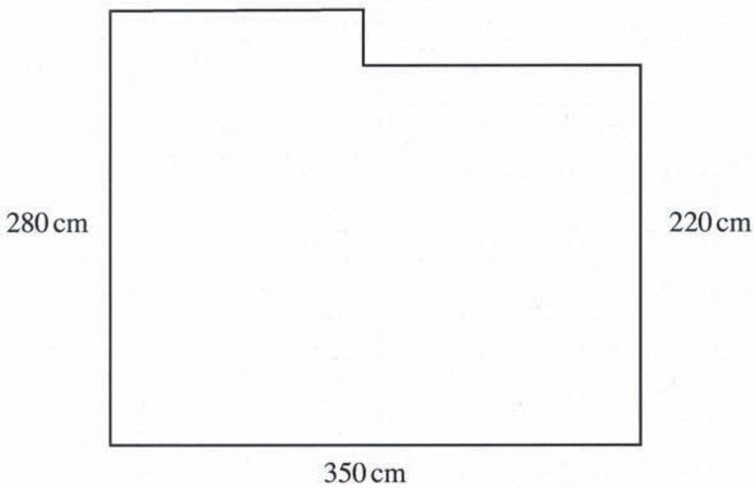
Summer 2009 Paper 2 Wales Pilot Higher Tier	**Mark**	**Comments**
Strategy, heights from 2 right-angled triangles	B1	
h_1=7.1sin28 or h_1=7.1cos62	M2	M1 for $\frac{h_1}{7.1}$ = sin 28 or $\frac{h_1}{7.1}$ = cos 62
h_1 = 3.3(32…cm)	A1	
Correct angle placement for second right angled triangle	B1	FT their 28 or 62
h_2=3.4cos28 or h_2=3.4sin62	M2	M1 for $\frac{h_2}{3.4}$ = cos 28 or $\frac{h_2}{3.4}$ = sin 62
h_2= 3.0(02…cm)	A1	
Shortest distance = 6.3 (cm)	B1	FT their h_1+h_2 only if both B marks awarded

WJEC Pilot GCSE Mathematics, Summer 2009, Higher Tier Paper 1, Question 7



Floor covering for a bathroom is sold in rectangular strips cut from rolls having widths of 3 m or 4 m.
The length of the strip may have any value, but its width must be either 3 m or 4 m.
The amount charged is based on the area of the rectangular strip.

Mr Blaggs measures his bathroom floor and draws a rough sketch, as shown below, to take to the shop.



Mr Blaggs wishes to buy a single strip of floor covering from one of the rolls.
He wants to spend as little as possible and to have no join in the floor covering.

Should Mr Blaggs buy a single strip of floor covering from the 3 m or 4 m wide roll?
You must show all your working to justify your answer.

Mark scheme

Summer 2009 Paper 1 Wales Pilot Higher Tier	Mark	Comments
Strategy, realising the rectangle is 280 by 350cm	M1	Not using the 220cm
Consistent understanding of cm and m	M1	Not mixing units
Strategy, 350cm of 3m roll AND 280 of 4m roll	M2	M1 for either, OR 400cm of 3m and 300cm of 4m
3.50×3 (=10.5m^2) AND 2.80×4 (=11.2m^2)	M2	Accept inconsistent units. M1 for either
Area from 3m roll $\times$ 6 OR Area from 4m roll $\times$ 5	M1	
(£) 63 AND (£)56	A2	CAO. A1 for either
Conclusion, 4m roll	A1	FT logic from their calculation provided at least M2 awarded

Quality of written communication (QWC)

In the new GCSE examinations, some questions will explicitly assess candidates' quality of written communication (QWC). This will include their mathematical communication used in answering specific questions. These questions, which will be clearly indicated on each question paper, will require learners to:

- ensure that the text is legible and that spelling, punctuation and grammar are accurate so that meaning is clear
- select and use a form and style of writing appropriate to the purpose and complexity of the subject matter
- organise information clearly and coherently, using specialist vocabulary where appropriate.

An exemplar question taken from the WJEC GCSE (linked pair) pilot in Applications of Mathematics is shown on the next page.

WJEC GCSE Pilot (Linked pair scheme) Applications of Mathematics

Unit 2: Financial, business and other applications
Higher Tier Specimen paper, Question 5(b)

You will be assessed on the quality of your written communication in this question.

Vikram reads a manufacturer's claim that

"a low energy light bulb lasts 20 times longer and uses $\frac{2}{3}$ of the electricity used by an ordinary light bulb."

An ordinary light bulb costs 49p and uses £3.30 of electricity over its lifetime.
A low energy light bulb costs £15.
By considering the period of the lifetime of one low energy light bulb, write a report explaining which type of light bulb offers better value for money for Vikram and by how much.

Mark scheme

Applications of Mathematics Specimen Paper Unit 2 Higher Tier	Mark	Comments
(b) 49p × 20 (= £9.80) **or** £3.30 × 20 (=£66)	M1	
£75.80 (for 20 ordinary bulbs)	A1	
Electricity for 1 low energy bulb cost = £2.20	B1	£2.20 seen
£2.20 × 20 = £44	M1	FT 20 × their electricity cost (provided < £3.30)
£59 (for 1 low energy)	A1	FT 20 × their elect cost + £15
Low energy bulb by £16.80	E1	FT conclusion **and** difference provided both M marks awarded
	QWC2	QWC2 Presents material in a coherent and logical manner, using acceptable mathematical form and with few, if any errors in spelling, punctuation and grammar. QWC 1 Presents materials in an organised manner, mainly using acceptable mathematical form, with some errors in spelling, punctuation and grammar. QWC 0 Evident weaknesses in organisation of material and errors in use of mathematical form and in spelling, punctuation and grammar.

The PISA survey of mathematical literacy

The Programme for International Student Assessment (PISA) is the world's largest international education survey. It is organised by the Organisation for Economic Co-operation and Development (OECD). In 2009, the PISA survey involved schools and learners across 65 countries. The survey focuses on the ability of 15-year-olds to use their skills and knowledge to address real-life challenges involving reading, mathematics and science.

The mathematics questions used in the PISA survey aim to assess learners' ability to use their mathematical skills and knowledge in different situations in adult life. This ability is referred to as mathematical literacy and is defined as ´*an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen*´.

The PISA surveys use a variety of types of question, from multiple choice and short answer questions to those requiring more extended responses. Some of the questions are purely mathematical but the vast majority are context-based problems that require the learner to engage with a situation and decide upon a method of solution.

The findings from Wales

In 2009, a total of 132 schools in Wales took part in the PISA survey. The results were extremely disappointing. In mathematics, Wales' mean score was significantly lower than the OECD average and the mean scores of each of the other UK nations. In addition, the mean score and Wales' international ranking fell when compared with the previous (2006) results. The performance distribution for Wales was heavily skewed towards the lower end, indicating underperformance at all levels.

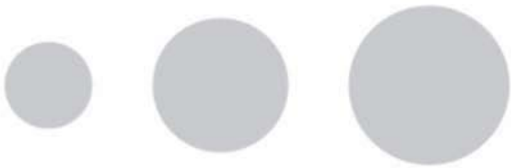
The implications for Wales

Clearly, action is needed to address these shortcomings. Some actions have already been taken, such as the changes in GCSE specifications described previously. Many of the revisions made at GCSE are in line with the PISA agenda – and if they are used to promote changes in teaching, then increases in performance should ensue. In order to inform teachers and learners of the style of the questions used in the PISA tests, some sample questions are reproduced on pages 14–18. Further sample questions that have been released by the OECD can be found on their website at www.oecd.org/dataoecd/47/23/41943106.pdf

Sample PISA questions

Coins

You are asked to design a new set of coins. All coins will be circular and coloured silver, but of different diameters.



Researchers have found out that an ideal coin system meets the following requirements.

- Diameters of coins should not be smaller than 15mm and not be larger than 45mm.
- Given a coin, the diameter of the next coin must be at least 30% larger.
- The minting machinery can only produce coins with diameters of a whole number of millimetres (e.g. 17mm is allowed, 17.3mm is not).

You are asked to design a set of coins that satisfy the above requirements.

You should start with a 15mm coin and your set should contain as many coins as possible. What would be the diameters of the coins in your set?

Mark scheme

Full credit: 15 – 20 – 26 – 34 – 45. It is possible that the response could be presented as actual drawings of the coins of the correct diameters.

Partial credit: Gives a set of coins that satisfy the three criteria, but not the set that contains as many coins as possible, e.g., 15 – 21 – 29 – 39, or 15 – 30 – 45
OR
The first three diameters correct, the last two incorrect (15 – 20 – 26 –)
OR
The first four diameters correct, the last one incorrect (15 – 20 – 26 – 34 –)

Student heights

In a mathematics class one day, the heights of all students were measured. The average height of boys was 160cm, and the average height of girls was 150cm. Alena was the tallest – her height was 180cm. Zdenek was the shortest – his height was 130cm.

Two students were absent from class that day, but they were in class the next day. Their heights were measured, and the averages were recalculated. Amazingly, the average height of the girls and the average height of the boys did not change.

Which of the following conclusions can be drawn from this information?

Circle 'Yes' or 'No' for each conclusion.

Conclusion	Can this conclusion be drawn?
Both students are girls.	Yes / No
One of the students is a boy and the other is a girl.	Yes / No
Both students are the same height.	Yes / No
The average height of the students did not change.	Yes / No
Zdenek is still the shortest.	Yes / No

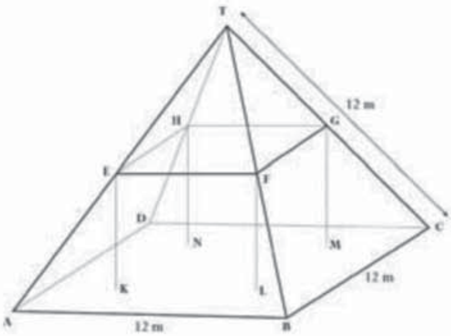
Mark scheme

Full credit: 'No' for all conclusions.

Farms

Here you see a photograph of a farmhouse with a roof in the shape of a pyramid.

Below is a student's mathematical model of the farmhouse **roof** with measurements added.



The attic floor, ABCD in the model, is a square. The beams that support the roof are the edges of a block (rectangular prism) EFGHKLMN. E is the middle of AT, F is the middle of BT, G is the middle of CT and H is the middle of DT. All the edges of the pyramid in the model have length 12m.

Calculate the area of the attic floor ABCD.
The area of the attic floor ABCD = __________m²

Calculate the length of EF, one of the horizontal edges of the block.
The length of EF = _________m

Mark scheme

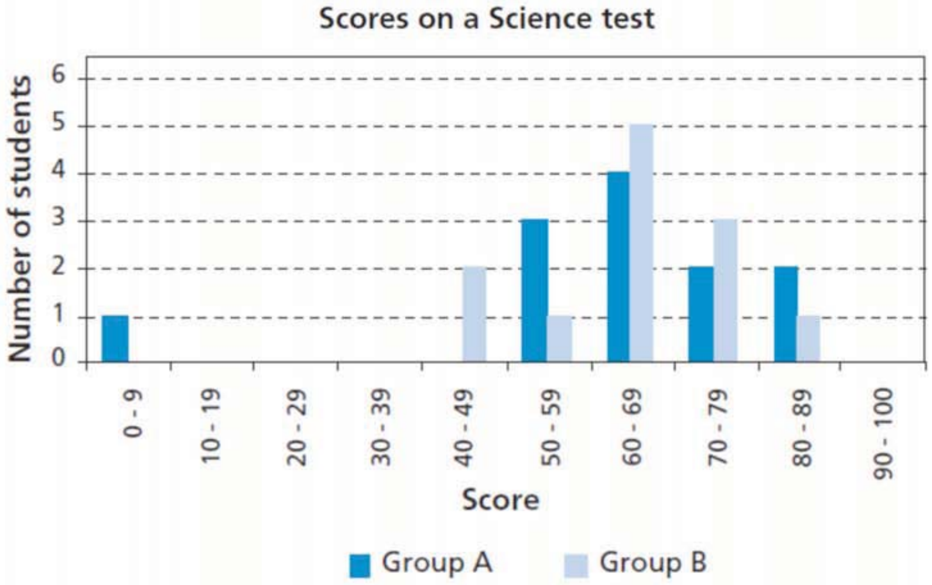
Full credit: 144 (unit already given)

Full credit: 6 (unit already given)

Test scores

The diagram below shows the results on a Science test for two groups, labelled as Group A and Group B.

The mean score for Group A is 62.0 and the mean for Group B is 64.5. Students pass this test when their score is 50 or above.



Looking at the diagram, the teacher claims that Group B did better than Group A in this test.

The students in group A don't agree with their teacher. They try to convince the teacher that Group B may not necessarily have done better.

Give one mathematical argument, using the graph, that the students in Group A could use.

Mark scheme

Full credit: One valid argument is given. Valid arguments could relate to the number of students passing, the disproportionate influence of the outlier, or the number of students with scores in the highest level.

- More students in Group A than in Group B passed the test.
- If you ignore the weakest Group A student, the students in Group A do better than those in Group B.
- More Group A students than Group B students scored 80 or over.

The best car

A car magazine uses a rating system to evaluate new cars, and gives the award of 'The Car of the Year' to the car with the highest total score. Five new cars are being evaluated, and their ratings are shown in the table.

Car	Safety Features (S)	Fuel Efficiency (F)	External Appearance (E)	Internal Fittings (T)
Ca	3	1	2	3
M2	2	2	2	2
Sp	3	1	3	2
N1	1	3	3	3
KK	3	2	3	2

The ratings are interpreted as follows:
3 points = Excellent
2 points = Good
1 point = Fair

To calculate the total score for a car, the car magazine uses the following rule, which is a weighted sum of the individual score points:

Total score = (3 x S) + F + E + T

Calculate the total score for car 'Ca'. Write your answer in the space below.

Total score for 'Ca: ....

The manufacturer of car 'Ca' thought the rule for the total score was unfair.

Write down a rule for calculating the total score so that car 'Ca' will be the winner.

Your rule should include all four of the variables, and you should write down your rule by filling in positive numbers in the four spaces in the equation below.

Total score = ...x S + ... x F + ... x E + ... x T.

Mark scheme

Full credit: 15 points.

Full credit: Correct rule that will make "Ca" the winner,
e.g. Total score = (2 x S) + F + E + (3 x T)

Implications for teachers

The actions needed to improve Wales' performance in future PISA mathematics surveys are similar to those required to prepare learners for the revised GCSEs in mathematics. Teachers need to stretch and challenge all their learners to reach their potential, and their more able learners, in particular, to reach the higher levels. This includes teaching higher-level mathematics content from the Range of the mathematics national curriculum, as well as applying lower-level content in context through the Skills. Applying mathematics in context can increase the demand considerably. For example, applying Pythagoras' theorem in a right-angled triangle to calculate the length of the hypotenuse is characteristic of Level 7, but determining whether a large wardrobe could be turned around in a small room or whether a pencil could fit inside a pencil box (where all the dimensions are given) pushes the demand to Level 8 and Exceptional Performance, respectively.

In order to allow learners to provide evidence of higher-level skills, they must be given freedom to make decisions, to select methods and to investigate within mathematics. Adopting a problem-solving approach to teaching and learning will broaden learners' mathematical experiences and enhance their learning opportunities. Taken from the Key Stage 3 Programme of Study, examples of opportunities that should be provided for learners to make decisions while solving problems include:

- *select . . . the mathematics, resources, measuring instruments, units of measure, sequences of operation and methods of computation needed to solve problems*
- *identify what further information or data may be required in order to pursue a particular line of enquiry . . .*
- *develop and use their own mathematical strategies and ideas . . .*
- *select, trial and evaluate a variety of possible approaches . . .*
- *. . . make conjectures and hypotheses, design methods to test them, and analyse results to see whether they are valid . . .*

To promote the development of learners' problem-solving skills, some GCSE questions will expect learners to make their own decisions. This will involve more than simply choosing the units to use for an answer, selecting the scales to adopt for the axes of a graph, or determining the intervals to use when classifying data. This will also require learners to draw their own diagrams or enhance one that is given, to devise their own approaches to tackle an unfamiliar problem, and to reflect on the accuracy or limitations of their solutions.

The level descriptions contain ‘signposts’ to the demand of each level; these are just examples of the associated demand. Many aspects of the mathematics curriculum are not mentioned at all within the level descriptions.

Teachers need to provide a rich experience for their learners, enhancing the programme of study by ensuring variety in their approach. Questions and tasks that are set should involve learners in collaborating to think their way through unfamiliar contexts and interesting situations, as well as consolidating their skills and knowledge in more familiar scenarios. In providing this variety of approach, teachers should take opportunities, whenever they arise, to drip-feed new ideas through lesson starters, consolidation activities, and everyday lessons, and to challenge learners’ thinking by frequently asking, ‘What if . . . ?’. In this way, learners will become used to thinking through new situations and will begin to appreciate the power and potential of mathematics.

If teachers invest time in stretching and challenging their learners, and extend their expectations of their ability then, in time, their learners will be able to apply their acquired toolkit of mathematical skills to tackle extended and open-ended problems. The tasks and activities within this booklet provide some ideas for teachers to use with their learners, while a wealth of further ideas can be found on the various websites that are referenced on pages 82–83. These ideas should be crystallised into activities that can be integrated into schemes of work. Such activities should become part of the normal approach to teaching and learning rather than bolted on as ’extras’.

Level descriptions

Level 7

Pupils justify their generalisations, arguments or solutions, consider alternative approaches and appreciate the difference between mathematical explanation and experimental evidence. They examine critically and justify their choice of mathematical presentation. In making estimates, they round to one significant figure and multiply and divide mentally. They understand the effects of multiplying and dividing by numbers between 0 and 1, and calculate proportional changes. They solve numerical problems with numbers of any size, using a calculator efficiently and appropriately. They describe in symbols the next term or nth term of a sequence with a quadratic rule. They use algebraic and graphical methods to solve simultaneous linear equations in two variables and solve simple inequalities. They use Pythagoras' theorem in two dimensions, calculate lengths, areas and volumes in plane shapes and right prisms, and enlarge shapes by a fractional scale factor. They appreciate the imprecision of measurement, and use compound measures such as speed. They specify and test hypotheses, taking account of bias. They analyse data to determine modal class and estimate the mean, median and range of sets of grouped data. They use measures of average and range to compare distributions, and draw a line of best fit on a scatter diagram by inspection. They use relative frequency as an estimate of probability and use this to compare outcomes of experiments.

Level 8

Pupils develop and follow alternative approaches, reflecting on their own lines of enquiry and using a range of mathematical techniques. They examine and discuss generalisations or solutions they have reached. They convey mathematical or statistical meaning through precise and consistent use of symbols. They solve problems involving calculating with the extended number system, including powers, roots and standard form. They manipulate algebraic formulae, equations and expressions. They solve inequalities in two variables. They sketch and interpret graphs of linear, quadratic, cubic and reciprocal functions, and graphs that model real situations. They understand congruence and mathematical similarity, and use sine, cosine and tangent in right-angled triangles. They interpret and construct cumulative frequency tables and diagrams. They compare distributions and make inferences, using estimates of the median and inter-quartile range. They solve problems using the probability of a compound event.

Exceptional Performance

Pupils give reasons for the choices they make when investigating within mathematics. They use mathematical language and symbols effectively in presenting a convincing reasoned argument, including mathematical justification. They express general laws in symbolic form. They solve problems using intersections and gradients of graphs. They use, generate and interpret graphs based on trigonometric functions. They solve problems in two and three dimensions using Pythagoras' theorem and trigonometric ratios. They calculate lengths of circular arcs, areas of sectors, surface areas of cylinders, and volumes of cones and spheres. They interpret and construct histograms. They understand how different sample sizes may affect the reliability of conclusions. They recognise when and how to use conditional probability.

Section 2

Examples of learners' work exemplifying higher-order level characteristics

Examples of learners' work exemplifying higher-order level characteristics

This section contains examples of learners' work within Key Stage 3 that demonstrate characteristics of Levels 7, 8 and Exceptional Performance. The work was collected from schools across Wales in 2009 and 2010. Each example is a learner's response to a mathematical task, and is accompanied by a commentary that aims to identify the characteristics of Level 7, Level 8 and Exceptional Performance inherent in the work.

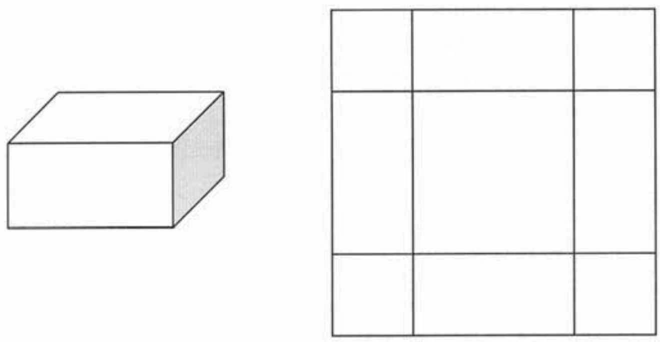
At the end of each example of a learner's work, a 'Way forward' section is included to provide feedback on how the work could have been improved and/or 'next steps' for the learner. In some instances, these 'next steps' include ideas for how work could be developed further either within Key Stage 3 or later in Key Stage 4.

Fruit farm cartons

This task is the classic 'max box' problem set in a context. It involves the construction of a three-dimensional container from its two-dimensional net. Learners were asked to find the dimensions necessary for the container to hold as much fruit as possible.

Fruit Farm Cartons

A "Pick Your Own" fruit farm requires open-topped boxes for the fruit pickers to carry their fruit in.



Squares of cardboard of side 18cm are used. A square is cut away from each corner and the four flaps are then folded up to make the open box.

How much should be cut away from each corner if the box is to hold as much fruit as possible?

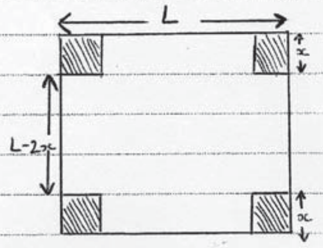
What if the original square of cardboard was a different size?
Can you find a general solution?

Bethan's work

Frut Farm Cartons

Aim: The aim of this investigation is to find the maximum volume of a container.

Design/Strategy: I will use squares of various lengths and cut away larger and larger corners. I will find the volume of each container and put the results in a table.



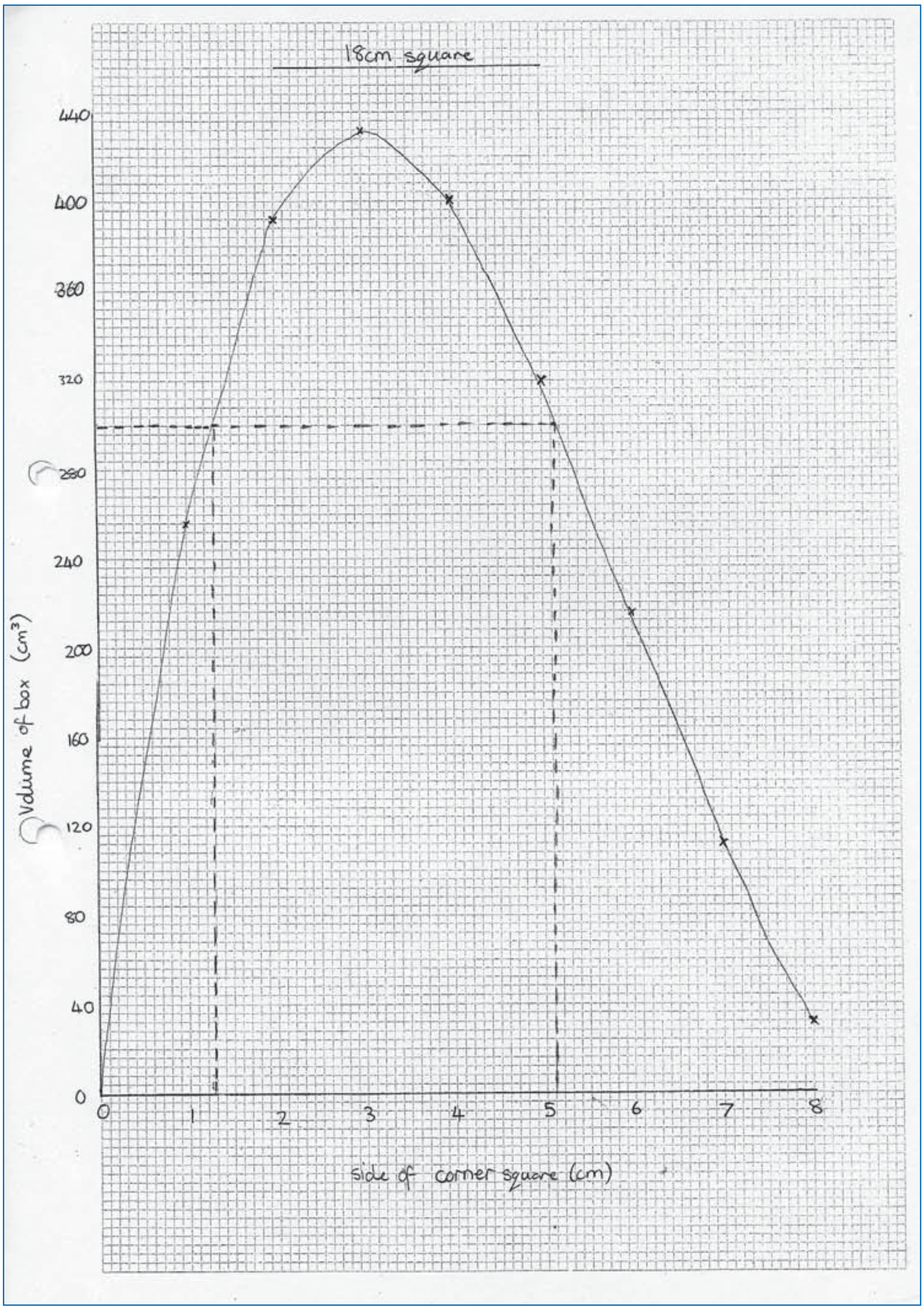
The volume of each container is $l \times b \times h$

$$V = (L-2x) \times (L-2x) \times (x)$$

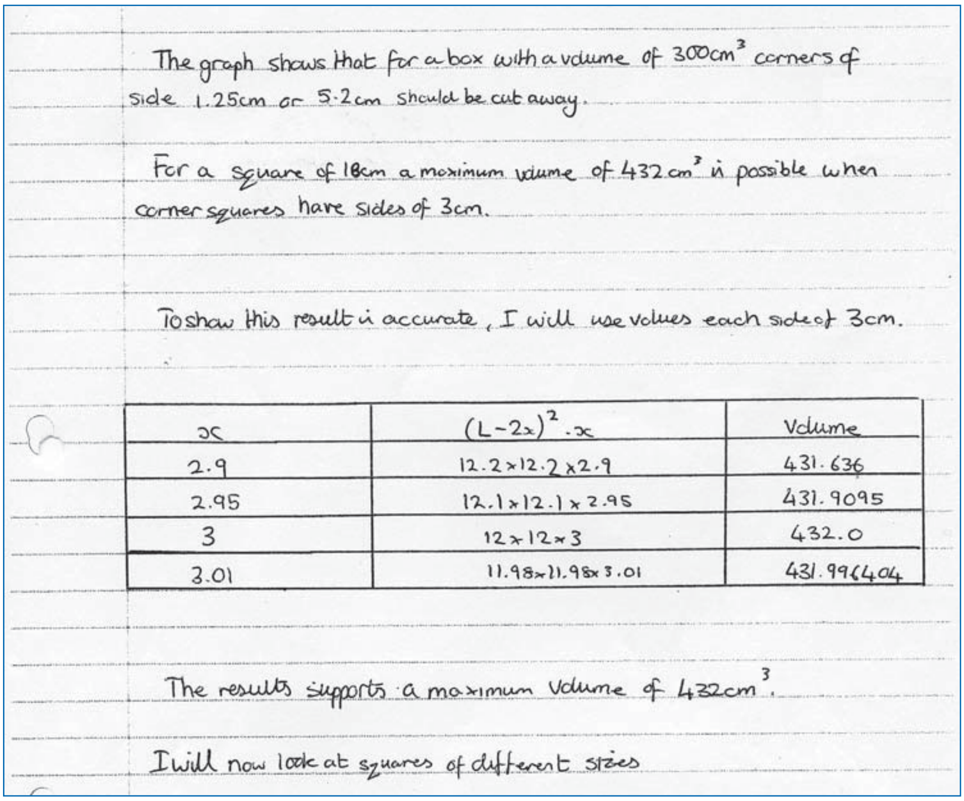
$$V = x(L-2x)^2$$

18 cm square

Length of corner cut away (cm)	Dimensions of the box (cm)		Volume of box (cm^3)
	Length/Width	Height	
1	16	1	256
2	14	2	392
3	12	3	432 (Max volume)
4	10	4	400
5	8	5	320
6	6	6	216
7	4	7	112
8	2	8	32



18cm square
440
400
360
320
280
240
200
160
120
80
40
0
Volume of box (cm³)
0
1
2
3
4
5
6
7
8
side of corner square (cm)


The graph shows that for a box with a volume of 300cm³ corners of side 1.25cm or 5.2cm should be cut away.

For a square of 18cm a maximum volume of 432cm³ is possible when corner squares have sides of 3cm.

To show this result is accurate, I will use values each side of 3cm.

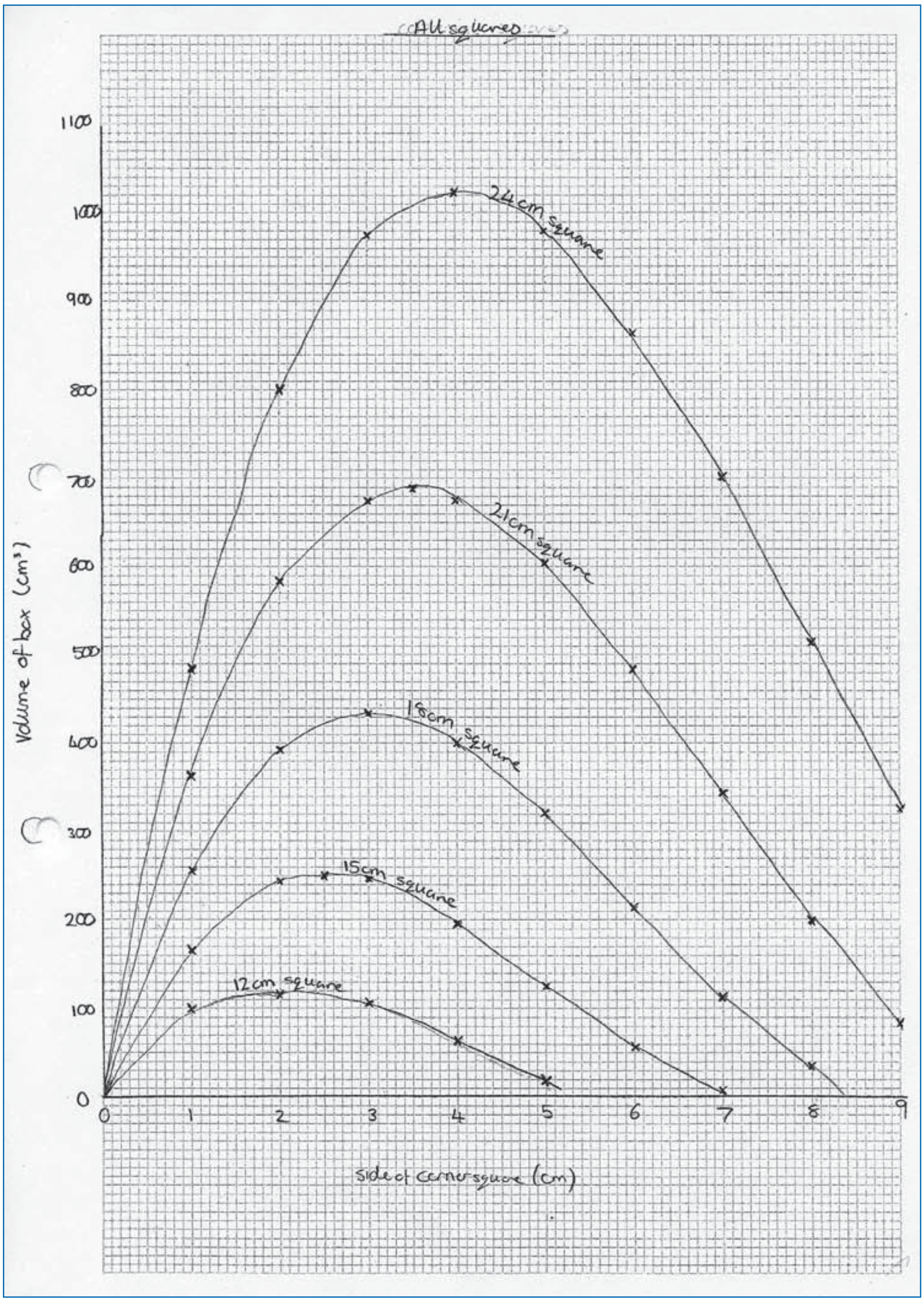
x	$(L-2x)^2 . x$	Volume
2.9	12.2×12.2×2.9	431.636
2.95	12.1×12.1×2.95	431.9095
3	12×12×3	432.0
3.01	11.98×11.98×3.01	431.996404

The results supports a maximum volume of 432cm³.

I will now look at squares of different sizes

Bethan introduced algebraic notation, labelled her diagram of a net, and wrote and simplified an algebraic formula for the volume of the open box formed. After methodically considering different sizes of squares to cut from each corner, she tabulated and plotted her results, before concluding that the maximum volume occurred when squares of side 3cm were cut from the corners. She went on to check her solution by using her derived algebraic formula to consider sizes just above and below 3cm.

Bethan shows a sound understanding of the different mathematical representations of the relationships that exist between the variables, including tabulated numerical results, graphs and algebraic formulae. Throughout her work, Bethan *conveys mathematical meaning through precise and consistent use of symbols,* which is characteristic of Level 8.



All squares
24cm square
21cm square
18cm square
15cm square
12cm square
Volume of box (cm³)
side of corner square (cm)
1100
1000
900
800
700
600
500
400
300
200
100
0
0
1
2
3
4
5
6
7
8
9

Length of square (cm)	12	15	18	21	24
maximum volume (cm^3)	128	250	432	686	1024
length of corner cut away (cm)	2	2.5	3	3.5	4

$12 \div 6 = 2$ $\quad 15 \div 6 = 2.5$ $\quad 18 \div 6 = 3$ $\quad 21 \div 6 = 3.5$ $\quad 24 \div 6 = 4$

$L \div 6 = x$ (for maximum volume).

Using my results, I suggest that the maximum volume of a container is when the corner square is cut away is 1/6 of the length of the original volume.

After solving the problem posed, Bethan extended her exploration by considering squares of cardboard of various sizes. She plotted and tabulated all her results together, before suggesting that the maximum volume occurs when the length of the squares to be cut away is one-sixth of the original length. The use of the word 'suggest' here indicates a recognition that this result is based on a small number of cases only and signifies an *appreciation of the difference between mathematical explanation and experimental evidence*, which is characteristic of Level 7.

The maximum volume can be estimated by using the following expression as shown:

$$V = x(L-2x)^2$$

$$x = \frac{L}{6}$$

$$V = \frac{L}{6}\left(L - \frac{L}{3}\right)^2$$

$$V = \frac{L}{6}\left(L - \frac{L}{3}\right)\left(L - \frac{L}{3}\right)$$

$$V = \frac{L}{6}\left[L^2 - \frac{L^2}{3} - \frac{L^2}{3} + \frac{L^2}{9}\right]$$

$$V = \frac{L^3}{6} - \frac{L^3}{18} - \frac{L^3}{18} + \frac{L^3}{54}$$

$$V = \frac{9L^3 - 3L^3 - 3L^3 + L^3}{54}$$

$$V = \frac{4L^3}{54}$$

$$V = \frac{2L^3}{27}$$

Had she written $L - \frac{L}{3}$ as $\frac{2L}{3}$, Bethan could have simplified her subsequent working in deriving her expression for the maximum volume: $V = \frac{2L^3}{27}$. Either way, the *manipulation of algebraic expressions* involved is characteristic of Level 8.

I will now draw a graph of maximum volume against powers of Length L

L	12	15	18	21	24
L^2	144	225	324	441	576
L^3	1728	3375	5832	9261	13824
Maximum Volume	128	250	432	686	1024

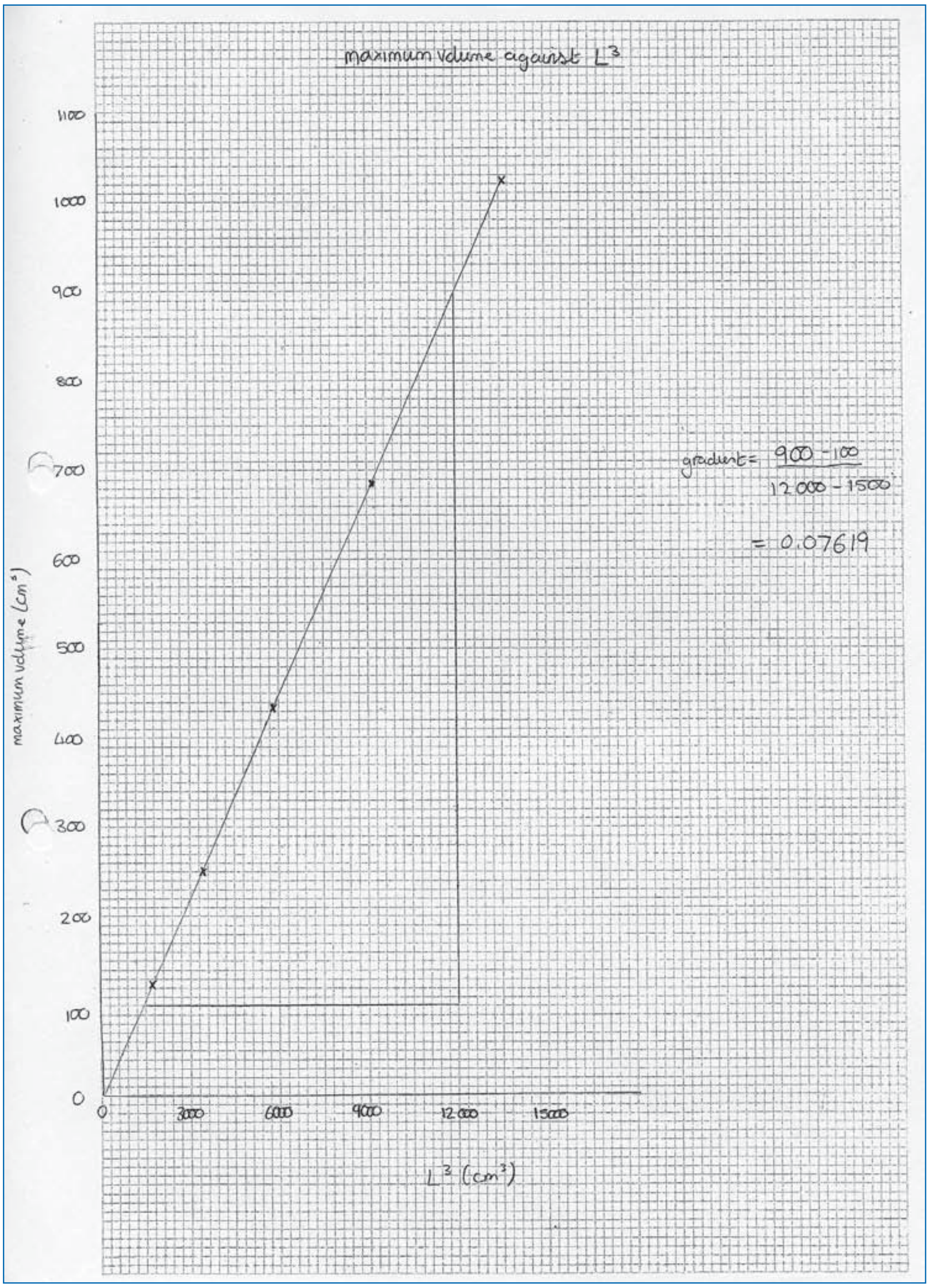
From my results, the graph of maximum volume and L^3 is a straight line through the origin

The gradient of the line $= \frac{900 - 100}{12000 - 1500} = 0.07619$

This gives $V = 0.07619 \times L^3$

This is a close approximation to $V = \frac{2}{27} L^3$ or $V = 0.0741 \times L^3$

Bethan constructed the graph of the maximum volume against L^3 and confirmed that the gradient of the straight line produced is approximately $\frac{2}{27}$. *Solving problems using . . . gradients of graphs* is characteristic of Exceptional Performance, although her work could be extended here by explaining clearly the significance of her findings.


maximum volume against L^3
maximum volume (cm^3)
L^3 (cm^3)
gradient = $\frac{900 - 100}{12000 - 1500}$
= 0.07619
0
100
200
300
400
500
600
700
800
900
1000
1100
3000
6000
9000
12 000
15000

Links to level descriptions

Characteristics of the **Level 7 description** include:

- *consider alternative approaches and appreciate the difference between mathematical explanation and experimental evidence.*

Characteristics of the **Level 8 description** include:

- *develop and follow alternative approaches, reflecting on their own lines of enquiry and using a range of mathematical techniques*
- *examine . . . solutions they have reached*
- *convey mathematical . . . meaning through precise and consistent use of symbols*
- *manipulate algebraic formulae . . . and expressions*
- *sketch and interpret graphs of linear, quadratic, cubic . . . functions.*

Characteristics of the **Exceptional Performance description** include:

- *solve problems using . . . gradients of graphs.*

Way forward

Making use of graph-plotting software would remove some of the repetitive nature of the work for Bethan. She could be asked to consider optimisation problems involving more complex shapes. Asking her to maximise the volume of a cylinder, cone, pyramid or hemisphere would lead to higher-level work on measures – for example, finding the maximum volume of a cone made from a sector of a circle of a given radius.

Alternatively, Bethan could be challenged to work on problems involving optimisation that are set in completely different contexts, such as those that involve maximising profit or minimising costs. Another alternative approach would be to introduce a design aspect to the task, such as designing a container to hold a given number of tennis balls.

UK coins

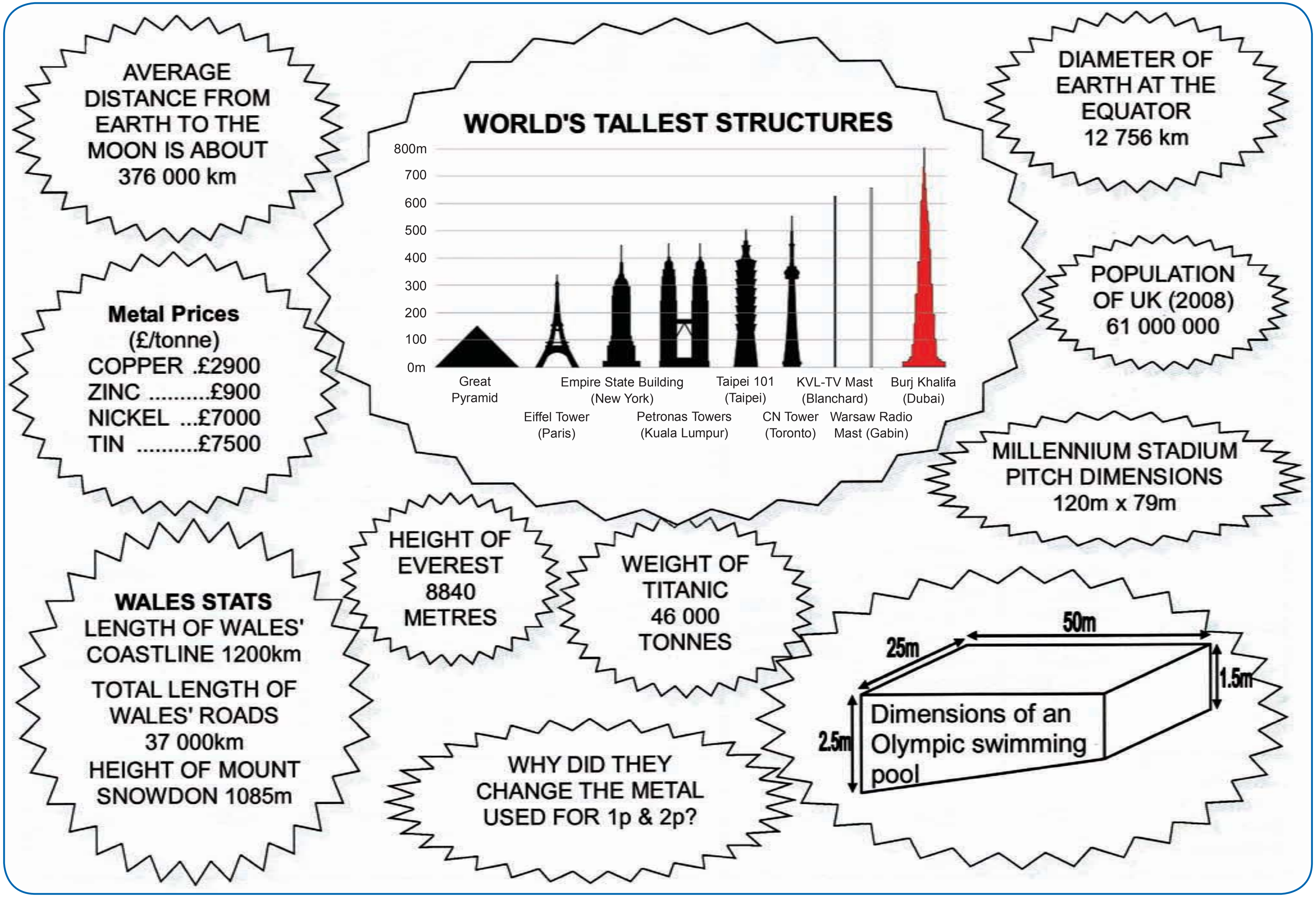
Learners were provided with two data sheets – one giving the specifications of UK coins in current circulation and the other providing a collection of useful data. For each coin, the specification gives details of its diameter, mass, thickness and composition, as well as giving the number in circulation (in 2008). After a class discussion session in which various suggestions were made for aspects that could be explored, learners were encouraged to choose their own investigation on which to work.

Investigate the number of coins in circulation in the UK. Use the information on the data sheets to help you describe the amount of coins in different ways.

UK COINS

UK COIN SPECIFICATIONS

Diameter	20.3mm	25.9mm	18.0mm	24.5mm	21.4mm	27.3mm	22.5mm	28.4mm
Weight	3.56g	7.12g	3.25g	6.5g	5.0g	8.0g	9.5 g	12.0g
Thickness	Bronze: 1.52mm Copper-plated steel: 1.65mm	Bronze: 1.85mm Copper-plated steel: 2.03mm	1.7mm	1.85mm	1.7mm	1.78mm	3.15mm	2.50mm
Composition	< Sept 92 Bronze 97% copper 2.5% zinc 0.5% tin >Sept 92 Copper-plated steel	< Sept 92 Bronze 97% copper 2.5% zinc 0.5% tin >Sept 92 Copper-plated steel	Cupro-nickel 75% copper 25% nickel	Cupro-nickel 75% copper 25% nickel	Cupro-nickel 84% copper 16% nickel	Cupro-nickel 75% copper 25% nickel	Nickel-Brass 70% copper 5.5% nickel 24.5% zinc	Nickel-Brass & Cupro-nickel 76% copper 14% nickel 10% zinc
Coins in Circulation in 2008 (millions)	10,576	6421	3659	1587	2190	769	1452	268



AVERAGE DISTANCE FROM EARTH TO THE MOON IS ABOUT 376 000 km
WORLD'S TALLEST STRUCTURES
800m
700
600
500
400
300
200
100
0m
Great Pyramid
Eiffel Tower (Paris)
Empire State Building (New York)
Petronas Towers (Kuala Lumpur)
Taipei 101 (Taipei)
CN Tower (Toronto)
KVL-TV Mast (Blanchard)
Warsaw Radio Mast (Gabin)
Burj Khalifa (Dubai)
DIAMETER OF EARTH AT THE EQUATOR 12 756 km
Metal Prices (£/tonne)
COPPER .£2900
ZINC £900
NICKEL ...£7000
TIN £7500
POPULATION OF UK (2008) 61 000 000
MILLENNIUM STADIUM PITCH DIMENSIONS 120m x 79m
WALES STATS
LENGTH OF WALES' COASTLINE 1200km
TOTAL LENGTH OF WALES' ROADS 37 000km
HEIGHT OF MOUNT SNOWDON 1085m
HEIGHT OF EVEREST 8840 METRES
WEIGHT OF TITANIC 46 000 TONNES
WHY DID THEY CHANGE THE METAL USED FOR 1p & 2p?
50m
25m
1.5m
2.5m
Dimensions of an Olympic swimming pool

Catrin's work

I am going to:

- investigate why they changed the metal used to make 1p coins
- compare 1p and 2p coins
- estimate the total length of all the UK coins if they are laid edge to edge
- estimate the total height of all the UK coins.

Why did they change the metal used to make 1p coins?

I think they changed the metal because copper is too expensive.

One old coin 97% copper, 2.5% zinc, 0.5% tin
97% of 3.56g = 97 ÷ 100 × 3.56 = 3.4532 grams of copper
2.5% of 3.56g = 2.5 ÷ 100 × 3.56 = 0.089 grams of zinc
0.5% of 3.56g = 0.5 ÷ 100 × 3.56 = 0.0178 grams of tin

Copper is £2 900 per tonne
= 2 900 x 100 = 290 000p per tonne
= 290p per kg
= 0.29p per gram

3.4532g x 0.29p = 1.001428p so the copper is worth more than 1p on its own.

Zinc is £900 per tonne
= 900 x 100 = 90 000p per tonne
= 90p per kg
= 0.09p per gram

0.089 x 0.09p = 0.00801p

Tin is £7 500 per tonne
= 7 500 x 100 = 750 000p per tonne
= 750p per kg
= 0.75p per gram

0.0178 x 0.75 = 0.01335p

Total cost of metal in old 1p = 1.001428 + 0.00801 + 0.01335 = 1.022788p which is more than its face value.

Catrin demonstrates fluent *use of compound measures* (pounds per tonne and pence per gram) while *solving a numerical problem . . . , using a calculator efficiently and appropriately*, both of which are characteristic of Level 7.

What about the 2p coin?

Is the 2p coin exactly twice as big as the 1p coin?

1p coin measures 20.3mm in diameter, 1.65mm thick and weighs 3.56g
2p coin measures 25.9mm in diameter, 2.03mm thick and weighs 7.12g

I can see straight away that the 2p weighs double the 1p, so the volume should be double.
$V = \pi r^2 h$

Volume of 1p = $\pi \times 10.15^2 \times 1.65 = 534.0303031mm^3$

Volume of 2p = $\pi \times 12.95^2 \times 2.03 = 1\ 069.511472mm^3$

$534.03 \times 2 = 1\ 068.0606mm^2$

The 2p is slightly more than double the volume of the 1p – but this could be because the original measurements have been rounded.

The volume factor of 2p from 1p is 2

Area factor = $\sqrt{2} = 1.4142$

Scale factor = $\sqrt[3]{2} = 1.2599$

If the 1p coin was enlarged the dimensions should be:

diameter = $20.3 \times \sqrt[3]{2} = 25.576$

depth = $1.65 \times \sqrt[3]{2} = 2.079$

This shows that even though the 2p is exactly double the weight of the 1p it is not a similar shape. If it was a similar shape it should have a diameter of 25.6mm not 25.9 and be 2.08mm thick instead of 2.03mm. So the 2p is slightly thinner but with a bigger diameter than it should be if it was an enlarged 1p.

So using the information that the 2p coin weighs double the 1p coin then the metal would cost double the amount. So 2p coins would cost 2.045p to make.

This is only slightly more than the value of the coin, but what about all the 1p and 2p coins?

10 576 000 000 x 1.022788 ÷ 100 = £108 170 059

108 170 059 — 105 760 000 = £2 410 059 cost more than they are worth

6 421 000 000 x 2.045576 ÷ 100 = £131 346 435

£131 346 435 — £128 420 000 = £2 926 435 cost more than they are worth

If the metal in 1p and 2p coins was bronze, their value would be £5.3 million pounds more than their face value!

Catrin's solution to this problem involves *calculating volumes* of cylinders (coins), which is characteristic of Level 7, and *calculating with the extended number system, including powers and roots*, and *understanding mathematical similarity*, which are both characteristic of Level 8. Catrin incorrectly states that the 'area factor' is $\sqrt{2}$, but this does not affect her results as she correctly states that the scale factor is $\sqrt[3]{2}$.

What about all the coins?

I am going to estimate how many times around the world all the UK coins will go.

1p coins $$\frac{20.3\text{mm} \times 10\ 576\ 000\ 000}{1\ 000\ 000} \approx 20 \times 11\ 000 = 220\ 000\text{km}$$

2p coins $$\frac{25.9\text{mm} \times 6\ 421\ 000\ 000}{1\ 000\ 000} \approx 30 \times 6\ 000 = 180\ 000\text{km}$$

5p coins $$\frac{18\text{mm} \times 3\ 659\ 000\ 000}{1\ 000\ 000} \approx 20 \times 3\ 600 = 72\ 000\text{km}$$

10p coins $$\frac{24.5\text{mm} \times 1\ 587\ 000\ 000}{1\ 000\ 000} \approx 25 \times 1\ 600 = 40\ 000\text{km}$$

20p coins $$\frac{21.4\text{mm} \times 2\ 190\ 000\ 000}{1\ 000\ 000} \approx 20 \times 2\ 200 = 44\ 000\text{km}$$

50p coins $$\frac{27.3\text{mm} \times 769\ 000\ 000}{1\ 000\ 000} \approx 30 \times 800 = 24\ 000\text{km}$$

£1 coins $$\frac{22.5\text{mm} \times 1\ 452\ 000\ 000}{1\ 000\ 000} \approx 22 \times 1\ 500 = 33\ 000\text{km}$$

£2 coins $$\frac{28.4\text{mm} \times 268\ 000\ 000}{1\ 000\ 000} \approx 30 \times 300 = 9\ 000\text{km}$$

Total distance in 1 000km =
220 + 180 + 72 + 40 + 44 + 24 + 33 + 9 = 622

Estimated total distance 622 000km
Distance around earth ≈ 40 000km

622 ÷ 40 ≈ 600 ÷ 40 = 15

So all the coins in circulation would go around the world about 15 times.

What about the total height?

To get a rough idea of total height I am going to use a rough figure of 2mm for the thickness of all coins.

Estimate of total number of coins in billions =
11 + 6 + 4 + 2 + 2 + 1 + 1 + 0 = 27

Estimated total number of coins = 27 000 000 000

So estimated height = $\frac{27\ 000\ 000\ 000 \times 2}{1\ 000\ 000}$ = 54 000km

Distance to moon ≈ 376 000km

Fraction of distance to moon = $\frac{54\ 000}{376\ 000} \approx \frac{50\ 000}{400\ 000} = \frac{1}{8}$

So all the coins in a pile would reach one eighth of the way to the moon.

In making her estimates, Catrin *rounds to one significant figure and multiplies and divides mentally*, which is characteristic of Level 7.

Conclusions

- Old 1p and 2p coins would cost more to make than their face value.
- All the coins in circulation in the UK laid edge to edge would go around the world about 15 times or they could reach to the moon and most of the way back.
- If they were piled up they would reach about one eighth of the distance to the moon.

Links to level descriptions

Characteristics of the **Level 7 description** include:

- *in making estimates, round to one significant figure and multiply and divide mentally*
- *solve numerical problems with numbers of any size, using a calculator efficiently and appropriately*
- *. . . calculate lengths, areas and volumes in plane shapes and right prisms . . .*
- *. . . use compound measures . . .*

Characteristics of the **Level 8 description** include:

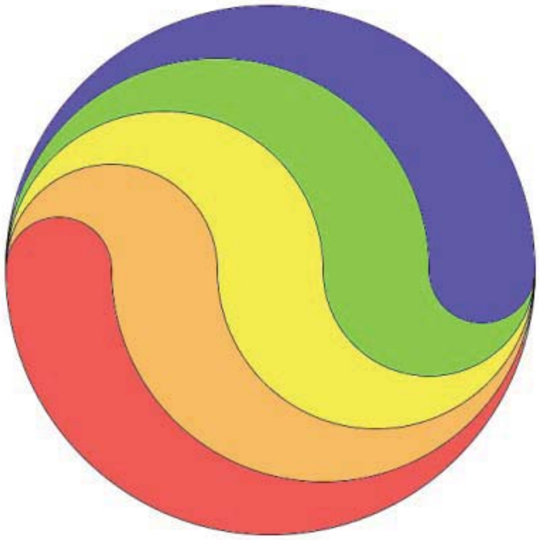
- *solve problems involving calculating with the extended number system, including powers, roots and standard form*
- *understand . . . mathematical similarity . . .*

Way forward

Catrin could be encouraged to set up a spreadsheet to compare her results with those for other coins, as this would reduce the need for repetitive calculations. Within this work, Catrin shows a misunderstanding of the ratio of areas of similar shapes, and although this did not affect the accuracy of her findings here, she could usefully be encouraged to tackle further problems that involve the ratio of lengths, areas and volumes of similar shapes.

Curvy areas

Take a look at the image below:



Can you see how the image was created?
Try to recreate it using a ruler and compasses.

Here are two smaller images created in a similar way.

Can you work out the proportion of each of the 3-, 4- and 5-colour circles which is shaded red?
Can you make any generalisations?
Can you prove your ideas?

What about the proportion which is shaded orange? . . . yellow?
Can you make any generalisations?
Can you prove your ideas?

(*Curvy areas* is taken from the Nrich website and can be accessed at: http://nrich.maths.org/6468)

Craig's work

3-colour circle

In the top semicircle the three semicircles are similar shapes.

Ratio of lengths = 1 : 2 : 3
Ratio of areas = 1 : 4 : 9

If the area of the small red semicircle is A,
then the area of the medium semicircle is $4A$
and the area of the large semicircle is $9A$.

So the areas of the three regions in the top semicircle are:
A
$4A - A = 3A$
$9A - 4A = 5A$

So the areas of the three shapes are:
Red shape: $A + 5A = 6A$
Orange shape: $3A + 3A = 6A$
Yellow shape: $5A + A = 6A$
All the shapes have equal areas so the red shape is $\frac{1}{3}$ of the area of the 3-colour circle.

Check

Area of small semicircle $= \dfrac{\pi r^2}{2}$

Area of medium semicircle $= \dfrac{\pi(2r)^2}{2} = \dfrac{4\pi r^2}{2} = 2\pi r^2$

Area of large semicircle $= \dfrac{\pi(3r)^2}{2} = \dfrac{9\pi r^2}{2}$

Area of coloured regions in the top semicircle are $\dfrac{\pi r^2}{2}$, $\dfrac{3\pi r^2}{2}$ and $\dfrac{5\pi r^2}{2}$

Area of red shape $= \dfrac{\pi r^2}{2} + \dfrac{5\pi r^2}{2} = \dfrac{6\pi r^2}{2} = 3\pi r^2$

Area of orange shape $= \dfrac{3\pi r^2}{2} + \dfrac{3\pi r^2}{2} = \dfrac{6\pi r^2}{2} = 3\pi r^2$

Area of yellow shape $= \dfrac{5\pi r^2}{2} + \dfrac{\pi r^2}{2} = \dfrac{6\pi r^2}{2} = 3\pi r^2$

So all of the shapes have equal areas.

4-colour circle

Ratio of lengths = 1 : 2 : 3 : 4
Ratio of areas = 1 : 4 : 9 : 16

If the area of the small red semicircle is A
then the area of the next semicircle is 4A
the area of the next semicircle is 9A and
the area of the largest semicircle is 16A

So the areas of the four regions in the top semicircle are:
A
4A – A = 3A
9A – 4A = 5A
16A – 9A = 7A

So the areas of the four shapes are:
Red shape: A + 7A = 8A
Orange shape: 3A + 5A = 8A
Yellow shape: 5A + 3A = 8A
Green shape: 7A + A = 8A

All the shapes have equal areas again so the red shape is $\dfrac{1}{4}$ of the area of the 4-colour circle.

It looks as if the red shape is $\dfrac{1}{n}$ of the area of the n-colour circle.

5-colour circle

This is the same again.
Ratio of areas = 1 : 4 : 9 : 16 : 25

The areas of the five regions in the top semicircle are:
A
4A – A = 3A
9A – 4A = 5A
16A – 9A = 7A
25A – 16A = 9A

So the areas of the five shapes are:
Red shape: A + 9A = 10A
Orange shape: 3A + 7A = 10A
Yellow shape: 5A + 5A = 10A
Green shape: 7A + 3A = 10A
Blue shape: 9A + A = 10A

All the shapes have equal areas again so the red shape is $\frac{1}{5}$ of the area of the 5-colour circle.

So again it looks as if the red shape is $\frac{1}{n}$ of the area of the n-colour circle.

n-colour circle

For a circle with n colours the areas of the regions in the top semicircle go up in odd numbers
A, 3A, 5A, 7A, 9A, etc.

Number of colours	**Area of largest region**
3	5A
4	7A
5	9A
n	2nA – A

In an n-colour circle the area of the largest region is 2nA – A

So the areas go A, 3A, 5A, 7A, 9A etc. up to 2nA – A
The 2nd largest area is 2nA – A – 2A = 2nA – 3A
The next region has area = 2nA – 3A – 2A = 2nA – 5A

So the areas of the n shapes are:
Red: A + 2nA – A = 2nA
Orange: 3A + 2nA – 3A = 2nA
Yellow: 5A + 2nA – 5A = 2nA
etc.

So all the areas are equal.

3-colour $\frac{6A}{3\times 6A}=\frac{1}{3}$

4-colour $\frac{8A}{4\times 8A}=\frac{1}{4}$

5-colour $\frac{10A}{5\times 10A}=\frac{1}{5}$

n-colour $\frac{2nA}{n\times 2nA}=\frac{1}{n}$

So the red shape and each of the other coloured shapes are $\frac{1}{n}$ of the area of the n-colour circle.

I am surprised to find that for each of the circles the areas of the coloured shapes that it is divided into are equal.

Craig deduced the proportion of the circles that were shaded red using a method based on *understanding mathematical similarity* in shapes, which is a characteristic of Level 8. He discovered that, within each of the circles, all the coloured regions are equal in area. His solution involved *manipulating algebraic formulae and expressions* accurately, which is also characteristic of Level 8. Further than this, Craig *used mathematical language and symbols effectively in presenting a convincing reasoned argument* to generalise his solution and to justify his generalisation, which is characteristic of Exceptional Performance.

Links to level descriptions

Characteristics of the **Level 8 description** include:

- *examine and discuss generalisations or solutions they have reached*
- *convey mathematical . . . meaning through precise and consistent use of symbols*
- *manipulate algebraic formulae, equations and expressions*
- *understand . . mathematical similarity . . .*

Characteristics of the **Exceptional Performance description** include:

- *give reasons for the choices they make when investigating within mathematics*
- *use mathematical language and symbols effectively in presenting a convincing reasoned argument, including mathematical justification.*

Way forward

Craig could be asked to extend his investigation by considering the perimeter of each of the coloured shapes. He presents his work very clearly and methodically. However, he may benefit from being exposed to some work on Euclidean geometry, for example, where engaging with formal proofs would be a useful challenge for him.

Square proof

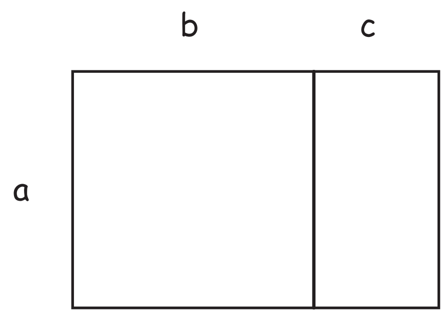
Two different rectangles are placed together, edge to edge, to form a large rectangle. The length of the perimeter of the large rectangle is $\frac{2}{3}$ of the total perimeter of the original two rectangles.
Prove that the final rectangle is in fact a square.

(This example is taken from the 2009 Olympiad Cayley Paper, published in *The UK Mathematics Trust Yearbook 2008–2009*, UKMT 2009)

Menna's work

If the two rectangles form a larger rectangle when they are put together, one of the sides of the two rectangles is the same length.

Call this length a and call the lengths of the other sides of the rectangles b and c.



Total perimeter of two rectangles = $2(a + b) + 2(a + c)$
$= 4a + 2b + 2c$

Perimeter of new rectangle = $2(a + b + c)$
$= 2a + 2b + 2c$

New perimeter = $\frac{2}{3}$ x original total perimeter

$2a + 2b + 2c = \frac{2}{3}(4a + 2b + 2c)$

$6a + 6b + 6c = 8a + 4b + 4c$

$2b + 2c = 2a$

$b + c = a$

So the sides of new rectangle are equal, $a = b + c$

So the rectangle is a square.

Menna has proved that the new rectangle formed by joining the two rectangles together is in fact a square. Her proof involves the *manipulation of algebraic equations and expressions* and *conveys mathematical meaning through precise and consistent use of symbols,* both of which are characteristic of Level 8. In fact, she goes further than this by *using mathematical language and symbols effectively in presenting a convincing reasoned argument*, which is characteristic of Exceptional Performance.

Links to level descriptions

Characteristics of the **Level 8 description** include:

- *convey mathematical . . . meaning through precise and consistent use of symbols*
- *manipulate algebraic formulae, equations and expressions.*

Characteristics of the **Exceptional Performance description** include:

- *use mathematical language and symbols effectively in presenting a convincing reasoned argument . . .*

Way forward

Menna is clearly ready to be stretched further, and an introduction to some more complex geometrical proofs could be a useful challenge for her.

An interesting problem

Sara has £500 to invest. The current interest rate is 6 per cent per year.

Sara knows that she would earn more under compound interest than simple interest. Investigate the interest she would receive under the two systems.

In particular, for how many years would she have to invest her money before the compound interest earned is:

- £100 more than the simple interest?
- £500 more than the simple interest?
- double the simple interest?

What if the amount invested and the interest rate were changed?

Investigate further . . .

Sara's work

Sara decided to draw up a spreadsheet to compare the interest received under the two systems, while many other learners chose to use a graphical approach. Her reason for using a spreadsheet was:

"I'll be able to see the effects of changing the amount invested or the interest rate straightaway and without starting all over again."

She produced several drafts before she arrived at the spreadsheet shown on page 50, which indicates her responses to the three questions posed:

(a) 11 years (b) 20 years (c) 23 years

Investment	£ 500
Rate	6 %

Year	Simple Interest	Amount	Total Simple Interest	Compound Interest	Amount	Total Compound Interest	CI – SI	CI ÷ SI	
1	30	530	30	30	530	30	0	1	
2	30	560	60	31.8	561.8	61.8	1.8	1.03	
3	30	590	90	33.71	595.51	95.51	5.51	1.06	
4	30	620	120	35.73	631.24	131.24	11.24	1.09	
5	30	650	150	37.87	669.11	169.11	19.11	1.13	
6	30	680	180	40.15	709.26	209.26	29.26	1.16	
7	30	710	210	42.56	751.82	251.82	41.82	1.2	
8	30	740	240	45.11	796.92	296.92	56.92	1.24	
9	30	770	270	47.82	844.74	344.74	74.74	1.28	
10	30	800	300	50.68	895.42	395.42	95.42	1.32	
11	30	830	330	53.73	949.15	449.15	119.15	1.36	**> £100 more**
12	30	860	360	56.95	1006.1	506.1	146.1	1.41	
13	30	890	390	60.37	1066.46	566.46	176.46	1.45	
14	30	920	420	63.99	1130.45	630.45	210.45	1.5	
15	30	950	450	67.83	1198.28	698.28	248.28	1.55	
16	30	980	480	71.9	1270.18	770.18	290.18	1.6	
17	30	1010	510	76.21	1346.39	846.39	336.39	1.66	
18	30	1040	540	80.78	1427.17	927.17	387.17	1.72	
19	30	1070	570	85.63	1512.8	1012.8	442.8	1.78	
20	30	1100	600	90.77	1603.57	1103.57	503.57	1.84	**> £500 more**
21	30	1130	630	96.21	1699.78	1199.78	569.78	1.9	
22	30	1160	660	101.99	1801.77	1301.77	641.77	1.97	
23	30	1190	690	108.11	1909.87	1409.87	719.87	2.04	**> double**
24	30	1220	720	114.59	2024.47	1524.47	804.47	2.12	
25	30	1250	750	121.47	2145.94	1645.94	895.94	2.19	

After using her spreadsheet to investigate further, Sara came to her conclusion.

If more money is invested or if the interest rate is higher then the difference between the compound interest and the simple interest will be bigger. So getting £100 more and £500 more will happen earlier.

If less money is invested or if the interest rate is lower then the difference will be smaller. So it will take longer before the same differences happen.

If the interest rate stays the same the compound interest will be double the simple interest after the same time.

If the interest rate goes up it will be double sooner and if the rate goes down it will take longer for it to be double.

The amount invested will not change the time for it to double if the interest rate is kept the same.

In drafting the spreadsheet for this investigation, Sara shows that she can *manipulate algebraic formulae and expressions*, while *using symbols in a precise and consistent manner,* which is characteristic of work at Level 8. Sara clearly understands how to solve problems related to the calculation of compound interest. Also, she is able to *present a reasoned argument* based on her findings and give *reasons for the choices she made in investigating* this problem. These are all characteristic of Exceptional Performance.

Links to level descriptions

Characteristics of the **Level 8 description** include:

- *convey mathematical . . . meaning through precise and consistent use of symbols*
- *manipulate algebraic formulae . . . and expressions.*

Characteristics of the **Exceptional Performance description** include:

- *give reasons for the choices they make when investigating within mathematics*
- *use mathematical language and symbols effectively in presenting a convincing reasoned argument . .*

Way forward

Sara could be asked to tackle more complex problems set in the world of finance involving the use of a spreadsheet, even though she has shown her skills in this area to be well developed already.

The hundred square

Learners were asked to investigate patterns in squares drawn on a hundred square.

Anwen's work

The Hundred Square.

In this investigation I am going to find out the total of numbers inside a square drawn on a hundred square.

2 x 2 squares

First of all I will try some numbers to see what I notice.

7	8
17	18

Total = 50

I've noticed that the top and bottom numbers end with the same numbers.

But you can't have a 9, 19, 29 etc. square because they don't fit on the grid.
Here are some more squares.

21	22
31	32

Total = 106

52	53
62	63

Total = 230

I've noticed that if you add 10 to the top row numbers you get the bottom row numbers.
Now I'll put my results in a table

Number in top left.	Total for square
7	50
21	106
52	230

Now I'm going to try to find a rule which connects the total for the square with the number in the top left.
To help me do this I'll find some more results.

1	2
11	12

Total = 26

2	3
12	13

Total = 30

3	4
13	14

Total = 34

4	5
14	15

Total = 38

Number in top left	Total for square	
1	26	
		4
2	30	
		4
3	34	
		4
4	38	

Totals are going up by 4, so rule will have 4n in it.
4n is 4, 8, 12, 16 so you need an extra 22.

So, if t is the total and n is the top left number, then the rule is

$t = 4n + 22$

Anwen *finds a rule for the nth term of a sequence where the rule is linear*, which is characteristic of Level 6.

Now I'm going to test my rule using the numbers from my first three squares.

When n is 7 $t = 4 \times 7 + 22$
$= 28 + 22$
$= 50$ ✓

when n is 21 $t = 4 \times 21 + 22$
$= 84 + 22$
$= 106$ ✓

when n is 52 $t = 4 \times 52 + 22$
$= 208 + 22$
$= 230$ ✓

So the rule works.

Why does my rule work?

If n is the top left number, then the square will be

n	n+1
n+10	n+11

So $t = n + n+1 + n+10 + n+11$
$= 4n + 22$

So my rule is correct.

After checking it, Anwen was able to *justify her generalised rule*, which is characteristic of Level 7.

3 x 3 Squares

Now I'll look at bigger squares, starting with 3 x 3 squares.

1	2	3
11	12	13
21	22	23

Total = 108

2	3	4
12	13	14
22	23	24

Total = 117

3	4	5
13	14	15
23	24	25

Total = 126

Results.

n	t
1	108
2	117
3	126
4	135

(differences: 9, 9, 9)

Rule is

$t = 9n + 99$

n	n+1	n+2
n+10	n+11	n+12
n+20	n+21	n+22

$t = n + n+1 + n+2 + n+10 + n+11 + n+12 + n+20 + n+21 + n+22$

$= 9n + 99$ ✓

So my rule works.

My rule for a 2 × 2 square was $t = 4n + 22$ & for a 3 × 3 is $t = 9n + 99$.
I can't predict the total for a 4 × 4 square but I've noticed that the Last number is a multiple of 11.

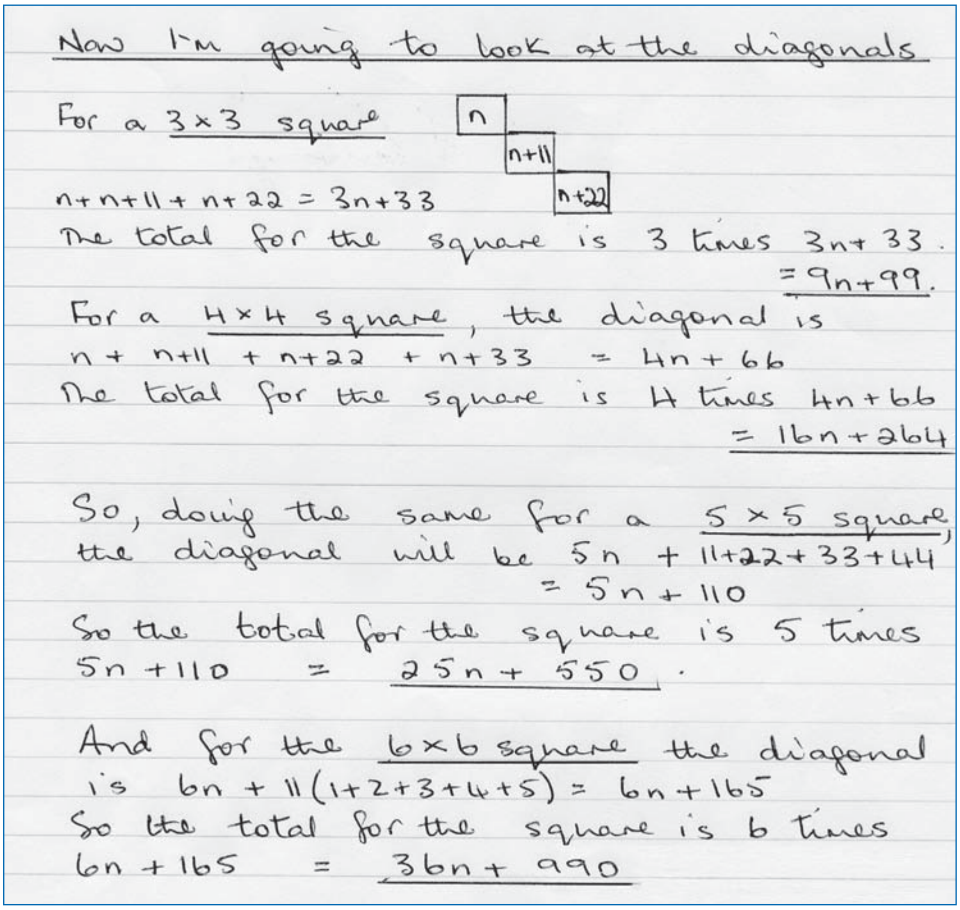
4 × 4 squares

The rule is $t = 16n + 264$.

Results so far

Square	Rule
2 × 2	$t = 4n + 22$
3 × 3	$t = 9n + 99$
4 × 4	$t = 16n + 264$ ← multiple of 11.

The numbers in front of n are square numbers, so I predict that for a 5 × 5 square $t = 25n + ?$



Now I'm going to look at the diagonals

For a 3 × 3 square

n+n+11+n+22 = 3n+33

The total for the square is 3 times 3n+33.

= 9n+99.

For a 4 × 4 square, the diagonal is

n + n+11 + n+22 + n+33 = 4n + 66

The total for the square is 4 times 4n+66

= 16n+264

So, doing the same for a 5 × 5 square, the diagonal will be 5n + 11+22+33+44

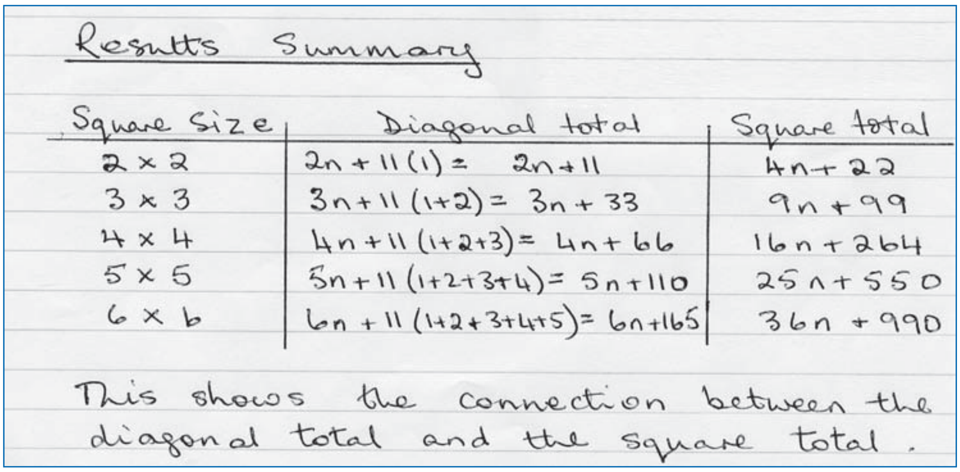
= 5n + 110

So the total for the square is 5 times 5n +110 = 25n + 550.

And for the 6×6 square the diagonal is 6n + 11(1+2+3+4+5) = 6n+165

So the total for the square is 6 times 6n + 165 = 36n + 990

Anwen experienced some difficulty in finding a generalised rule for a square of any size, so she considered the leading diagonal alone. She noticed that the total of the numbers in the leading diagonal of a square of side n could be multiplied by n to give the total of all the numbers in the square. *Reflecting on her line of enquiry, following an alternative approach and examining her generalisation* in this way are all characteristic of Level 8.



Results Summary

Square Size	Diagonal total	Square total
2 × 2	2n + 11(1) = 2n+11	4n+22
3 × 3	3n+11 (1+2) = 3n + 33	9n+99
4 × 4	4n+11 (1+2+3) = 4n+66	16n+264
5 × 5	5n+11 (1+2+3+4) = 5n+110	25n+550
6 × 6	6n + 11 (1+2+3+4+5) = 6n+165	36n +990

This shows the connection between the diagonal total and the square total.

Looking at my results, I think that in a p x p square the diagonal will be

$pn + 11(1+2+3+\ldots(p-1))$

and the square total will be

p times the diagonal

$t = [pn + 11(1+2+3+\ldots(p-1))] \times p$.

Anwen was able to derive a generalised rule for a square of any size. In doing this, she *described in symbols the nth term of a sequence with a quadratic rule*, which is characteristic of Level 7. However, the complexity of this rule, being a generalisation of a sequence of linear rules, is sufficient for the work to be characteristic of the demand associated with Level 8.

Links to level descriptions

Characteristics of the **Level 7 description** include:

- *justify their generalisations, arguments or solutions, consider alternative approaches . . .*
- *describe in symbols the next term or nth term of a sequence with a quadratic rule.*

Characteristics of the **Level 8 description** include:

- *develop and follow alternative approaches, reflecting on their own lines of enquiry and using a range of mathematical techniques*
- *examine and discuss generalisations or solutions they have reached.*

Way forward

Anwen could be asked to reflect on her rule for the total of the numbers in the leading diagonal of the 3 by 3 square, and to justify why this is one-third of the total of the numbers in the 3 by 3 square. She could be asked to simplify her formulae for the totals (in the leading diagonal and the squares), by recognising and using the sequence of triangular numbers (1+2+3+...). In addition, Anwen could be asked to investigate other relationships on a hundred square, for example, sums or differences within various shapes, or relationships within such shapes on different number grids.

Team competition

Teams A, B, C and D competed against each other once. The results table was as follows.

Team	Win	Draw	Loss	Goals for	Goals against
A	3	0	0	5	1
B	1	1	1	2	2
C	0	2	1	5	6
D	0	1	2	3	6

(a) Find (with proof) which team won in each of the six matches.

(b) Find (with proof) the scores in each of the six matches.

(This example is taken from the 2009 Olympiad Cayley Paper, published in *The UK Mathematics Trust Yearbook 2008–2009*, UKMT 2009.)

Akram's work

(a) I need to find the results of 6 matches.

A v B, A v C, A v D, B v C, B v D and C v D.
A won all 3 matches so A v B, A v C and A v D were all won by A.

C lost 1 match and drew the other 2 matches.
C lost to A, so B v C and C v D were draws.

B won 1 match, drew 1 match and lost 1 match.
B lost to A and drew against C, so must have won against D.

So the 6 match results were

A v B	A won	B v C	draw
A v C	A won	**B** v D	B won
A v D	A won	C v D	draw

(b) Now I need to find the scores of the 6 matches.

B won 1, drew 1 and lost 1, but only scored 2 goals altogether, so the draw against C must have been 0 – 0 or 1 – 1.
It can't have been a 2 – 2 draw as that doesn't leave any goals for their win over D.

B scored 2 goals altogether so their win over D must have been 1 – 0, 2 – 0 or 2 – 1.
B scored 2 goals and let 2 goals in so the only possible scores for B's matches are:

B v D	1 – 0 or	1 – 0 or	2 – 0 or	2 – 1
B v C	0 – 0	1 – 1	0 – 0	0 – 0
B v A	1 – 2	0 – 1	0 – 2	0 – 1
	2 – 2	2 – 2	2 – 2	2 – 2

C's score against A was 0 or 1 since only 1 goal was scored against A altogether. So C scored 0 or 1 against both A and B, and C scored 5 goals altogether so C's draw against D must be at least 3 – 3.

D only scored 3 goals altogether so C v D must be 3 – 3.

C scored 5 goals altogether, so must have scored 1 against A and 1 against B.
C drew against B so it must have been 1 – 1.
C had 6 goals scored against them altogether.
They drew 3 – 3 against D and 1 – 1 against B, so they must have lost 1 – 2 to A.
So C's scores were:

C v A	1 – 2
C v B	1 – 1
C v D	3 – 3
	5 – 6

If the score for B v C was 1 – 1 then B's other scores must have been 1 – 0 against D and 0 – 1 against A. (See the second column above.)

D drew 3 – 3 against C and lost both their other matches, without scoring any goals and letting in 6 goals altogether, so they must have lost 1 – 0 and 2 – 0. As they lost 1 – 0 against B (see above) they must have lost 2 – 0 against A.

The scores for the 6 matches were:

A v B	A won 1 – 0	B v C	draw 1 – 1
A v C	A won 2 – 1	B v D	B won 1 – 0
A v D	A won 2 – 0	C v D	draw 3 – 3

Akram's solution to this problem, particularly in part (b), which becomes surprisingly complicated, involves the *presentation of a convincing reasoned argument* and is characteristic of Exceptional Performance.

Links to level descriptions

Characteristics of the **Exceptional Performance description** include:

- *use mathematical language . . . effectively in presenting a convincing reasoned argument.*

Way forward

Akram has shown that he can think through a complex situation, and present his work methodically. He could be asked to engage with further problem-solving tasks, including those that involve more advanced mathematical content from the Range.

Cheesecake

Brief: You are an adviser who writes a column in *Nut Monthly* magazine. A reader sends in the following problem which is passed on to you by the editor. He wants the detailed reply to be published in next month's issue.

Dear Sir

I have a recipe for cheesecake as follows:

Base:	6 oz crushed digestive biscuits
	2 oz butter
	2 oz chopped nuts
Filling:	12 oz soft cheese
	2 oz sugar
	2 teaspoons cornflour

Grease a 7.5 inch diameter cake tin.
Cover the base of the tin with chopped nuts.
Melt the butter and mix with the crushed biscuits.
Spoon the mixture . . .

The recipe continues to explain how to make the cheesecake.

a) A 15-inch diameter cheesecake is to be made for a party. I need to find the quantity of chopped nuts needed to make the cheesecake for the party. Is it just a matter of doubling the quantity of this ingredient? I would appreciate a full explanation on this matter for future cheesecake making.

b) Individual cheesecakes are to be made using 0.5 oz of chopped nuts. Please suggest three different shapes (one of which must include a circle) and sizes of cake tins for making the individual cheesecakes. Please would you show all your working.

c) I need to put some decorative ribbon around the individual cheesecakes as suggested in b). Would you please suggest which shape would be the cheapest to decorate?

Yours faithfully

A.Cook

A Cook

David's work

Areas and Perimeters of circles

Dear Sir/Madam.

After reading your letter, I was determined to find the answer for you. I have set up three separate investigations that may solve your problem. For the people reading this in our magazine, these are the questions I aim to answer:-

Investigation 1 – If you double the diameter of a cheesecake is it just a case of doubling the amount of chopped nuts needed?

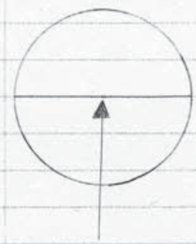
Investigation 2 – What other shapes of cheesecake can you have, were you only need to use 0.5oz of chopped nuts? What would the measurements of the different shapes of cheesecake tins be?

Investigation 3 – Which out of the three different shapes of cheesecake will be the cheapest to rap a ribbon around for decoration?

Investigation 1

I will start this investigation by drawing two diagrams to represent cheesecakes

cheesecake 1a



Diameter = 7.5 inches

cheesecake 1b

Diameter = 15 inches

I will now find the area of each circle using πr^2.

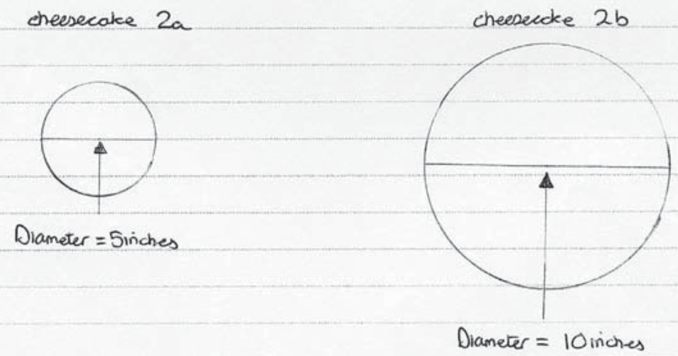
cheesecake 1a: $A = \pi r^2$
$= \pi(3.14) \times 3.75^2$
$= 44.15625$
$= 44.2$ sqins to 3 sf

cheesecake 1b: $A = \pi r^2$
$= 3.14 \times 7.5^2$
$= 176.625$
$= 177$ sqins to 3 sf

Using my answers I would suggest that if you double the diameter of a cheesecake you don't double the amount of ingredients needed.

<u>I will now show that my prediction is correct</u>

I am going to draw a diagram of a cheesecake with a diameter of 5 inches and draw a second diagram of a cheesecake with a diameter of 10 inches. To show that my prediction was correct the area of the cheesecake with the diameter of 10 inches needs to be different than the area of the other cheesecake ×2.



$A = \pi r^2$	$A = \pi r^2$
$= 3.14 \times 2.5^2$	$= 3.14 \times 5^2$
$= 19.625$	$= 78.5$
$= 19.6$ sq ins to 3s.f	$= 78.5$ sq ins to 3sf

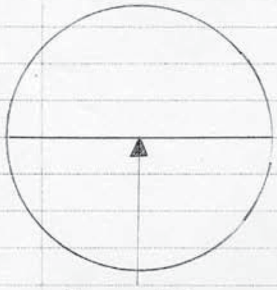
Cheesecake 1		Cheesecake 2
Diameter		Diameter
5 inches	×2 →	10 inches

cheesecake 1		cheesecake 2
Area of cheesecake		Area of cheesecake
19.6 sq ins	×4 →	78.5 sq ins

As you can see from my diagram it is not a case of doubling the amount of chopped nuts if you double the diameter of the cheesecake.
If you double the diameter of the cheesecake you actually times the amount of ingredients by 4.

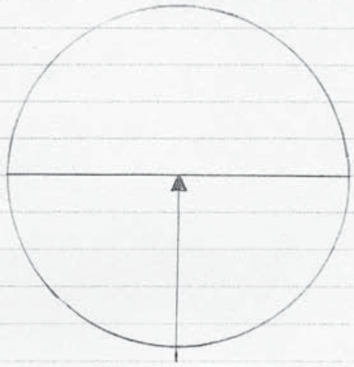
To convince you completely that my prediction is correct, I am going to use a cheesecake with a diameter of 15 inches from the beginning of the investigation and double it to 30 inches and see if my prediction still works.

cheesecake 3a



Diameter = 15 inches

cheesecake 3b



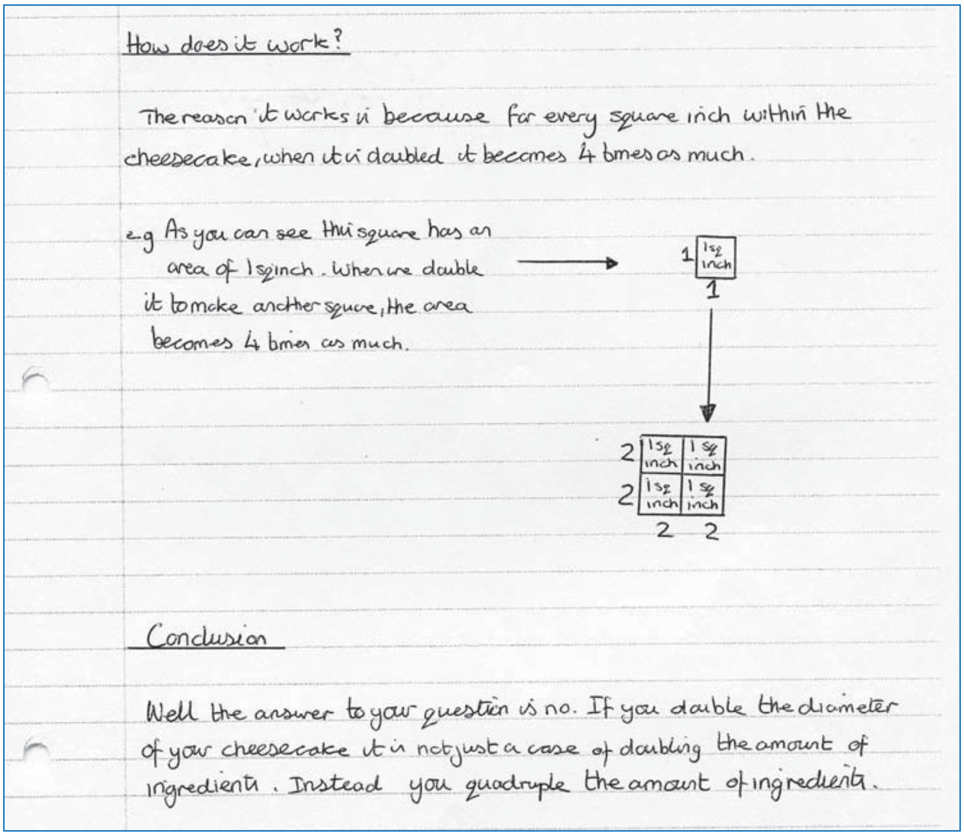
Diameter = 30 inches

$A = \pi r^2$
$= 3.14 \times 7.5^2$
$= 176.625$
$= 177$ sq ins to 3 s.f

$A = \pi r^2$
$= 3.14 \times 15^2$
$= 706.5$
$= 707$ sq in to 3 s.f

cheesecake 3a — Diameter 15 inches → ×2 → cheesecake 3b — Diameter 30 inches

cheesecake 3a — Area of cheesecake 177 sq ins → ×4 → cheesecake 3b — Area of cheesecake 707 sq ins


How does it work?

The reason it works is because for every square inch within the cheesecake, when it is doubled it becomes 4 times as much.

e.g As you can see this square has an area of 1 sq inch. When we double it to make another square, the area becomes 4 times as much.

Conclusion

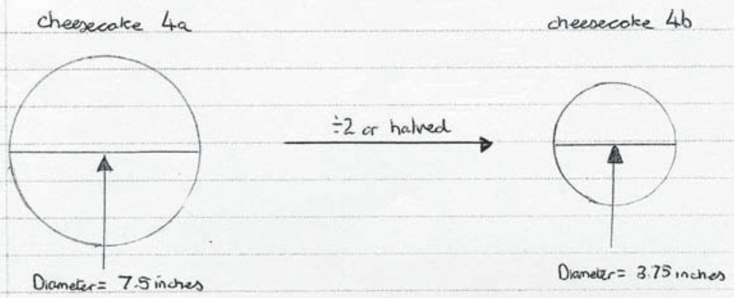
Well the answer to your question is no. If you double the diameter of your cheesecake it is not just a case of doubling the amount of ingredients. Instead you quadruple the amount of ingredients.

David discovered that doubling the diameter of a circle results in its area increasing four-fold, which necessitates a quadrupling of the ingredients. He provided a mathematical justification for this result by considering the effect of doubling the sides of a one-inch square. Although the significance of the 2s around the diagram is ambiguous, David's intention is clear, and this *justification of his solution* is characteristic of Level 7.

Had David used algebraic notation to formalise his mathematical justification by proving his conjecture that when the diameter of a circle is doubled, its area is quadrupled, this would have been characteristic of Exceptional Performance.

Investigation 2

Firstly, I need to find the diameter of the cheesecake with 0.5oz of chopped nuts in the ingredients.



2oz of chopped nuts → ÷4 → 0.5oz of chopped nuts

The reason that the diameter is halved but the amount of chopped nuts is divided by 4 is because in the first investigation, I found out that if I doubled the diameter of a cheesecake I quadrupled the amount of ingrediants well this time if I half the diameter I divide by 4 the amount of ingrediants.

Secondly, I need to find the area of cheesecake 4b

$A = \pi r^2$

$= 3.14 \times 1.875^2$

$= 11.0390$

$= 11.0$ sq ins to 3 s.f.

David used the result he reached in the first part to reason that a quarter of the ingredients would be sufficient for a circular cheesecake of half the diameter. This *examination* and use *of* a derived *generalisation or solution* is characteristic of Level 8.

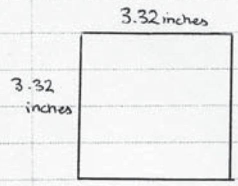
Now that I have found the area of the cheesecake, I know what area the two other shaped cheesecake must be.

One of the shapes I am going to investigate will be a square for the inexperienced chef and for the more advanced, I will investigate a cheesecake shaped like a star.

The square

To find the measurements of the square, what I have to do is quite simple. The area of the square is 11 and the two sides will equal the same number, this means if I square root 11, I will be given the measurements.

$\sqrt{11} = 3.32$ inches.



I will now test my method to see if it is correct :-

area of square $= 3.32 \times 3.32$

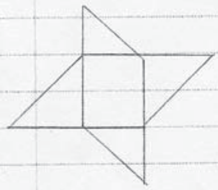
$= 11.0224$

$= 11.0$ sq ins to 3s.f

My method was correct; to make a square cheesecake you need a cake tin with the measurements 3.32 inches by 3.32 inches.

The Star

To find the measurements of the star, the formula is rather complicated.
Firstly, I need to split the star into five different shapes.



Now that I have split the star into different shapes I need to give each shape an area.
I am going to give the square an area of 4 and to find the area of each right angled triangle I need to divide the remaining area (7 sq ins) by 4.

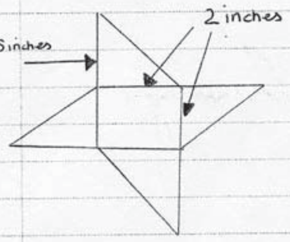
Area of triangle = 7÷4
= 1.75 sq ins

To find the measurements of the square I need to square root the area.
= √4
= 2 inches

Now that I have the measurements of the square, I also have the length of the base of each triangle. I know this because each base is connected to each side of the square. All I need to find now is the height of each triangle.

What number when multiplied by 2 and then divided by 2 = 1.75 sq ins, well its obviously 1.75 inches.

I will now test my method to see if I was correct :-



Area of square = 2 × 2
= 4 sq ins

Area of 1 triangle = $\frac{\text{base} \times \text{height}}{2}$
= $\frac{2 \times 1.75}{2}$
= 1.75 sq ins

Area of 4 triangles = 1.75 × 4
= 7 sq ins

Area of star = 7+4
= 11 sq ins

My method was correct; to make a cheesecake shaped like a star you need a cake tin with the same measurements as the diagram above.

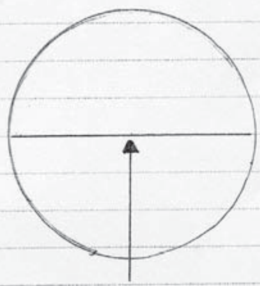
Conclusion

During this investigation, I have used the circle shaped cheesecake to help me find two other shaped cheesecakes with the same area. I hope I have shown enough working and the shapes are to your liking.

Investigation 3

To answer your question I need to find the perimeter of each shape.

Shape 1

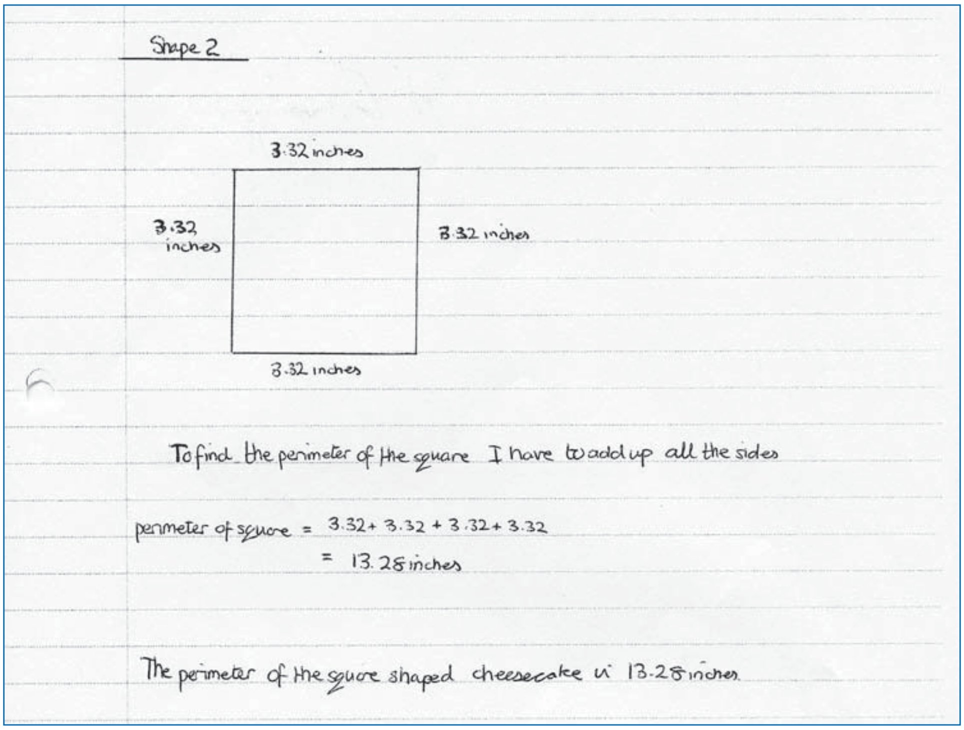


Diameter = 3.75 inches

To find the perimeter of a circle I have to multiply the diameter by π "3.14"

perimeter of circle = π × d
= 3.14 × 3.75
= 11.775 inches

The perimeter of the circle shaped cheesecake is 11.775 inches



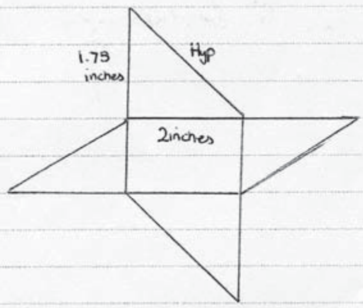
Shape 2

To find the perimeter of the square I have to add up all the sides

perimeter of square = 3.32 + 3.32 + 3.32 + 3.32
= 13.28 inches

The perimeter of the square shaped cheesecake is 13.28 inches.

Shape 3



I currently only have the height of each triangle; to be able to find the perimeter, I need the hypotenuse (Hyp) of each triangle as well. To find this I need to use Pythagoras's theorem.

Pythagoras's theorem

Pythagoras's theorem is a formula that finds the hypotenuse of a right angle triangle using the length of the two other sides. In this formula the hypotenuse is going to be classed as "H"

$$Hyp^2 = side^2 + side^2$$
$$H^2 = 2^2 + 1.75^2$$
$$H^2 = 7.0625$$
$$H = \sqrt{7.0625}$$
$$H = 2.6575364$$
$H = 2.66$ inches to 3s.f

Now, to find the perimeter of the whole star, I need to add 1.75inches and 2.66inches together and multiply by 4.

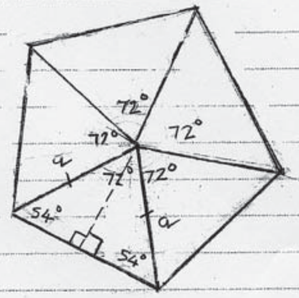
Perimeter of star $= (1.75 + 2.66) \times 4$
$= 4.41 \times 4$
$= 17.64$ inches

The perimeter of the star shaped cheesecake is 17.64 inches

David found the perimeter of the star shape, using *Pythagoras' theorem in two dimensions*, which is characteristic of Level 7.

Shape 4

A regular pentagon is made up of five isosceles triangles. The angles at the centre a



72° each i.e 360 ÷ 5

If the area is 11inches^2 then 1 triangle has an area of 11 ÷ 5 which is 2.2inches^2.

I will use the area formula for non right angled triangles.

$\frac{1}{2}ab\sin 72 = 2.2$

$ab\sin 72 = 4.4$

$a = \sqrt{\frac{4.4}{\sin 72}}$

$a = 2.1509$

To find the perimeter of the shape, I will use cosine.

$\cos 54 = \frac{adj}{hyp}$

adj = cos54 × 2.1509

adj = 1.2643

side = 1.2643 × 2

side = 2.5286 inches

side = 2.53inches to 3s.f

∴, the perimeter for the ribbon will be 2.5 × 5 = 12.5 inches.

David extended the task by investigating a cake tin with a pentagonal cross-section. In calculating the perimeter of the pentagon, he used trigonometry, including the area formula:

$$A = \frac{1}{2}ab\sin C$$

.

Solving this problem in two dimensions using trigonometric ratios is characteristic of Exceptional Performance, as it goes beyond the use of trigonometry in a right-angled triangle.

Summary of results;

shape	ribbon length
circle	11.775 inches
square	13.28 inches
star	17.64 inches
pentagon	12.5 inches

From the table, the cheapest shape for the ribbon is the circle.

Conclusion

During this investigation, I have found the perimeter of each cheesecake to find out which one would be the cheapest to decorate with a ribbon. I have discovered that the simple circle shaped cheesecake is the cheapest to decorate and the most expensive is the star.

I hope all three investigations were up to your standards and I have solved all your problems.

Yours faithfully.

Links to level descriptions

Characteristics of the **Level 7 description** include:

- *justify their generalisations, arguments or solutions . . .*
- *solve numerical problems . . . using a calculator efficiently and appropriately*
- *use Pythagoras' theorem in two dimensions . . .*

Characteristics of the **Level 8 description** include:

- *examine and discuss generalisations or solutions they have reached*
- *. . . use sine, cosine and tangent in right-angled triangles.*

Characteristics of the **Exceptional Performance description** include:

- *solve problems in two and three dimensions using Pythagoras' theorem and trigonometric ratios.*

Way forward

David could be asked to extend his work by attempting to justify his conclusion that a circular cake tin would be the cheapest shape to decorate with ribbon. His understanding of the ratio of lengths and areas of similar figures could be further developed and consolidated, before being extended to consider the ratio of lengths, areas and volumes of similar solids.

Painted cube

Sam's work

Painted cube investigation

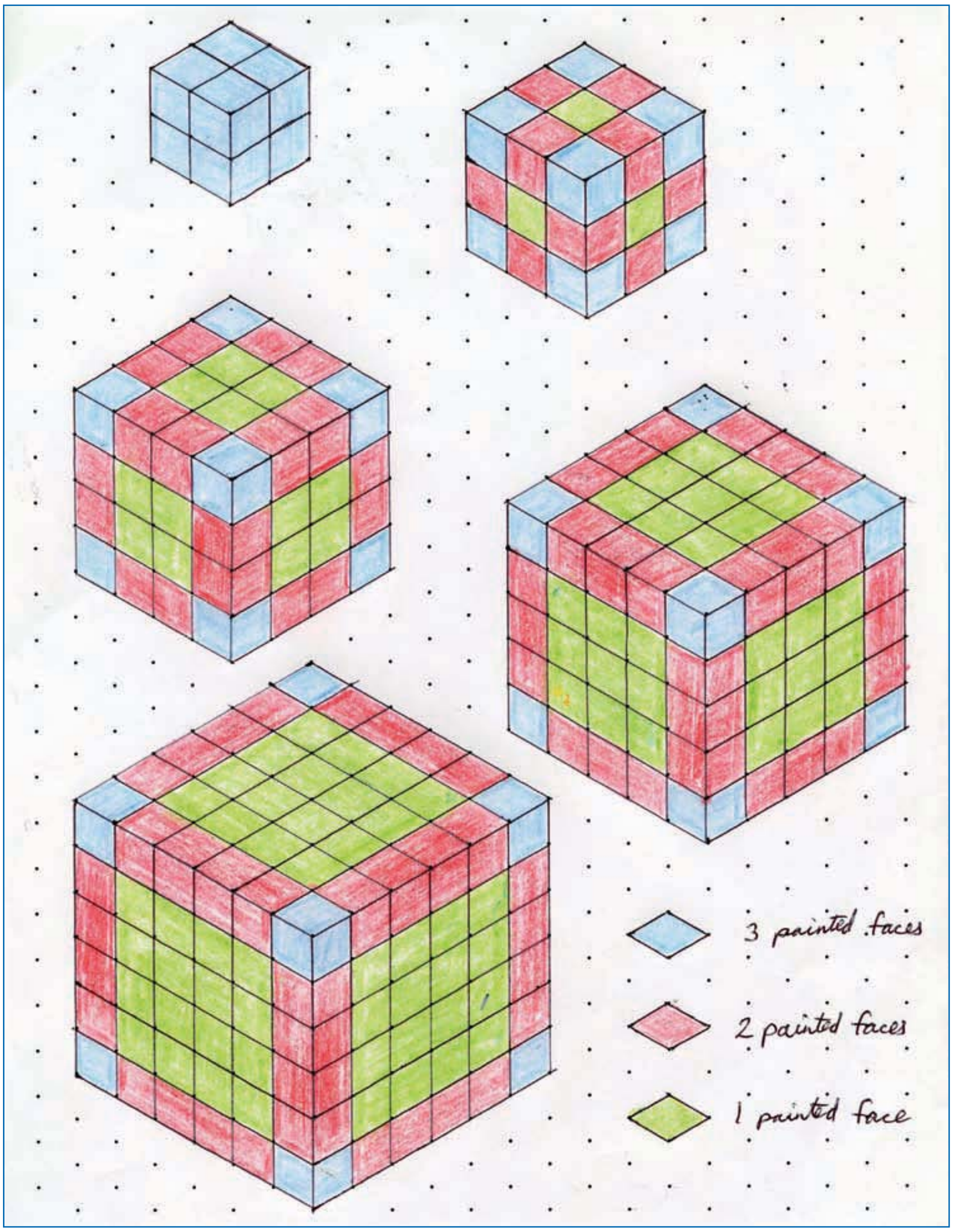
The problem:

I'm going to investigate the different patterns that appear when painting the faces of different sized cubes, then I'll dismantle the big cube into little cubes and then I'll be able to see how many faces of the little cubes have been painted.

My plan:

- To begin with I'll be drawing pictures of different sized cubes.
- I will then be recording the number of faces that have been coloured in a table.
- Afterwards I will look for patterns.
- To finish, I want to find formulas for a cube with sides n cm.

This is what I did to find patterns after drawing pictures of different sized cubes. I recorded my results in a table.



Length of cube (cm)	Number of cubes	Number of cubes with this number of painted faces			
		0	1	2	3
1	1	0	0	0	1
2	8	0	0	0	8
3	27	1	6	12	8
4	64	8	24	24	8
5	125	27	54	36	8

These are the patterns I have noticed:

- The number of cubes with 3 sides that have been painted is always 8 because a cube has 8 corner cubes at the 8 vertices – unless it's a single cube.
- To get the amount of cubes altogether we have to cube the length of the cube.
- Adding the number of cubes that have been painted always gives the total number of cubes, e.g. 8 + 24 + 24 + 8 = 64 which is the number of small cubes in a 4cm cube.
- The number of cubes with 2 faces that have been painted is equal to the length of the cube – 2 x 12

  For example, for the 5cm cube 5 – 2 = 3
  3 x 12 = 36

  36 cubes have 2 painted faces.
- The number of cubes with 0 faces painted is the length of the cube take away 2 then cubed.
- The number of cubes with 1 face painted is a multiple of 6.
- There are 0 cubes with 4 or 5 or 6 cubes painted because all cubes have 3 or more faces hidden inside the big cube.

Next I have found formulas for a cube with sides n cm.

Length of cube (cm)	Number of cubes	Number of cubes with this number of painted faces			
		0	1	2	3
n	n^3	$(n-2)^3$	$6(n-2)^2$	$12(n-2)$	8

I am now going to check that my formulas work for a 6 x 6 x 6 cube.
n = 6
Number of cubes with 3 painted faces = 8
Number of cubes with 2 painted faces = 12 x (6 – 2) = 12 x 4 = 48
Number of cubes with 1 painted face = $6 \times (6 - 2)^2 = 6 \times 16 = 96$
Number of cubes with 0 painted face = $(6 - 2)^3 = 4^3 = 64$
Check: $8 + 48 + 96 + 64 = 216 = 6^3$

Length of cube (cm)	Number of cubes	Number of cubes with this number of painted faces			
		0	1	2	3
6	216	64	96	48	8

Explanation

- The number of cubes with 3 painted faces is always 8 because it doesn't matter what size the cube is. It will always have 8 vertices if it's not a single cube.
- Multiply the number of cubes on one face that have one face painted by 6 because a cube has 6 faces.
- Multiply the number of cubes on one edge that have 2 faces painted by 12 because a cube has 12 edges.
- It's n – 2 because to get the length of the edge, we have to take away two corners.

I have now come to a conclusion. This is what I have found during this investigation.

Number of cubes with 3 painted faces = 8 cubes every time = 8 corners
Number of cubes with 2 painted faces = $12(n-2)$
Number of cubes with 1 painted face = $6(n-2)^2$
Number of cubes with 0 painted faces = $(n-2)^3$

I could do further investigating into this problem by exploring other shapes such as cuboids, prisms and pyramids. The easiest to investigate would be a cuboid as it is very similar to a cube.

- There would be 8 vertices again, therefore the number of cubes with 3 painted faces would still be 8, like the cube.
- It would still be x 6 because a cuboid has 6 faces — no matter what the size of the cuboid.
- It would still be x 12 because a cuboid has 12 edges.
- The length of the edges will be in the formula again as n – 2.

I have discovered a lot by doing this investigation.

Sam reviewed his strategies and suggested an alternative strategy he could have used for looking at cause and effect. This is a characteristic of the Level 8 description.

Sam systematically considers cubes of increasing side length, finding the numbers of small cubes with 0, 1, 2 and 3 painted faces in each case. He is able to *generalise his findings by finding formulae* and give some limited *justification for his formulae*, both of which are characteristic of Level 7.

Sam progressed to consider the same problem applied to cuboids but instead of working through some simple examples first, he attempted to generalise and incorrectly stated that the results would be the same as for cubes. There are similarities and he is correct about the corner cubes, with 3 painted faces. However, further thought is required in deriving formulae for the number of cubes with 0, 1 and 2 painted faces. Nonetheless, Sam's *precise and consistent use of symbols* in the work he has undertaken on cubes is characteristic of Level 8.

Links to level descriptions

Characteristics of the **Level 7 description** include:

- *justify their generalisations, arguments or solutions . . .*
- *describe in symbols the next term or nth term of a sequence with a quadratic rule.*

Characteristics of the **Level 8 description** include:

- *examine and discuss generalisations or solutions they have reached*
- *convey mathematical . . . meaning through precise and consistent use of symbols.*

Way forward

Sam could be asked to attempt to express his justifications for his generalised rules more clearly. He could be asked to show that the expressions for the number of cubes with 0, 1, 2 and 3 painted faces do sum to n^3, the total number of small cubes. Sam needs to be encouraged to reconsider this problem when applied to cuboids, and then to try to generalise his findings for a cuboid of any size.

Useful resources and websites

Resources

These materials were developed by or in conjunction with DfES.

Mathematics in the National Curriculum for Wales (Welsh Assembly Government, 2008)

Mathematics: Guidance for Key Stages 2 and 3 (Welsh Assembly Government, 2009)

Making the most of learning: Implementing the revised curriculum (Welsh Assembly Government, 2008)

Ensuring consistency in teacher assessment: Guidance for Key Stages 2 and 3 (Welsh Assembly Government, 2008)

A curriculum of opportunity: developing potential into performance (ACCAC, 2003)

Skills framework for 3 to 19-year-olds in Wales (Welsh Assembly Government, 2008)

Developing thinking and assessment for learning programme (Welsh Assembly Government):

- *Why develop thinking and assessment for learning in the classroom?*
- *How to develop thinking and assessment for learning in the classroom*
- Developing thinking and assessment for learning poster and leaflet.

All the above materials are available from the Welsh Government's website at www.wales.gov.uk/educationandskills

Aiming for Excellence: Developing thinking across the curriculum (BBC Cymru Wales, Estyn, Welsh Assembly Government, 2006) (Available from BBC Cymru Wales)

Websites

The websites listed below contain a wealth of ideas for further mathematical activity.

Association of Teachers of Mathematics (ATM): resources for teachers to download or use online in the classroom, designed to help develop a creative and thinking approach in mathematics learners.
www.atm.org.uk/resources/

Bowland Maths:
- innovative case study problems, each taking 3–5 lessons, designed to develop thinking, reasoning and problem-solving skills
- stand-alone assessment items, each of which takes less than one lesson to complete, and some no more than 20 minutes
- professional development materials to help teachers develop the skills needed for the case studies and for the revised programme of study.

www.bowlandmaths.org.uk/

CensusatSchool: free downloadable resources containing a variety of classroom activities, some directly related to the 2011 CensusAtSchool project.
www.censusatschool.org.uk/resources/

Cre8ate: downloadable resources involving the application of mathematics in interesting contexts.
www.cre8atemaths.org.uk/resources

GeoGebra: free dynamic software for learning and teaching that brings together geometry, algebra, statistics and calculus in one package.
www.geogebra.org/cms/

MathsNet: a range of games, puzzles, investigations and other activities.
www.mathsnet.net/

NGfL Cymru: a range of games and starter activities to develop mathematical and thinking skills.
www.ngfl-cymru.org.uk/eng/index-new.html

NRICH: numerous enrichment materials for learners aged 5 to 19 and their teachers.
http://nrich.maths.org/forteachers

Nuffield Applying Mathematical Processes (AMP): twenty activities accessible to all secondary learners, comprising abstract investigations and practical explorations set in realistic contexts.
www.nuffieldfoundation.org/about-applying-mathematical-processes-amp

SchoolsWorld TV: a comprehensive selection of mathematics programmes, online resources and useful web-links, formerly found on Teachers TV.
www.schoolsworld.tv/subjects/secondary/maths

TSM Resources: a wealth of internet resources and mathematical entertainment for classroom use, including links to sites within the UK and from across the world.
www.tsm-resources.com/mlink.html

United Kingdom Mathematics Trust: junior, intermediate and senior mathematics challenges to stretch more able learners.
www.mathcomp.leeds.ac.uk/

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Ysgol Botwnnog, Gwynedd
Ysgol Clywedog, Wrexham
Ysgol Glan Clwyd, Denbighshire.

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NRICH (www.nrich.maths.org)

'Mathematics sample tasks', in OECD, *Take the Test: Sample Questions from OECD's PISA Assessments,* OECD Publishing (2009), http://dx.doi.org/10.1787/9789246050815-4-en

The Royal Mint

United Kingdom Mathematics Trust (UKMT)
(www.mathcomp.leeds.ac.uk)

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