Key Stage 3
National Strategy

Guidance

Curriculum and Standards

Mathematics study modules

Consultant edition

Mathematics consultants and teachers

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Key Stage 3 National Strategy: Mathematics study modules

Introduction for teachers

ABOUT THE KEY STAGE 3 NATIONAL STRATEGY

The Key Stage 3 Strategy was founded in 2001. It focuses on four important principles:

- Expectations
  Establishing high expectations for all pupils and setting challenging targets for them to achieve
- Progression
  Strengthening the transition from Key Stage 2 to Key Stage 3 and ensuring progression in teaching and learning across Key Stage 3
- Engagement
  Promoting approaches to teaching and learning that engage and motivate pupils and demand their active participation
- Transformation
  Strengthening teaching through a programme of professional development and practical support

AIMS OF THESE STUDY MODULES

These ten *Mathematics study modules,* designed for an individual teacher or group of teachers, have been produced by the mathematics strand of the Key Stage 3 Strategy. They are intended for teachers who would like to reinforce, confirm and extend their knowledge of the Key Stage 3 mathematics curriculum and to develop their teaching skills. They are suitable for all teachers of mathematics in Key Stage 3, including supply teachers, trainee teachers, and those who would like to return to teaching.

The modules are based on a successful two-part course, *Planning and teaching mathematics*, that has been offered to practising teachers in all LEAs.

The aims of the modules are to:

- develop teachers' understanding of important aspects of the mathematics curriculum in Key Stage 3;
- strengthen teachers' planning and teaching of mathematics in Key Stage 3.

This guidance and introduction to the modules is intended for teachers who are planning to study them. Separate guidance is available for mentors.

CHOOSING WHICH MODULES TO STUDY

There is no need to study all the modules unless you want to. Each is designed as a self-contained unit of work. Choose those that most interest you and that will give you the most help. Descriptions of the ten modules are as follows:

Module 1 Approaches to calculation

- Discusses the development of calculation strategies from Key Stage 2 to Key Stage 3 to build on pupils' knowledge, skills and understanding
- Considers teaching methods to support the development of calculation strategies, using subtraction as a focus

Module 2 Using calculators

- Discusses when it is and when it is not appropriate for pupils to use a calculator
- Considers the skills that pupils need to use a calculator effectively and efficiently
- Discusses the progression of calculator skills and the implications for teaching

Module 3 Thinking about algebra

- Considers some stimulating activities for teaching an aspect of algebra, the simplification of algebraic expressions
- Discusses how the activities can also help to develop algebraic reasoning
- Considers how the activities might be incorporated in mathematics lessons

Module 4 Making links in algebra

- Considers the development of work on sequences, functions and graphs in Key Stage 3
- Considers links across different aspects of algebra and links with other strands of the mathematics curriculum
- Discusses the use of challenging activities to develop pupils' algebraic reasoning and the use of algebra in solving problems

Module 5 Geometrical reasoning 1

- Looks at approaches to developing pupils' visualisation and geometrical reasoning skills
- Considers progression towards geometric proof

Module 6 Geometrical reasoning 2

- Considers the connections between loci and constructions
- Discusses activities and resources to develop pupils' visual and geometrical reasoning skills

Module 7 Ratio and proportion 1

- Discusses ratio and proportion as a key mathematical idea, with applications across many aspects of Key Stage 3 mathematics and in other subjects
- Analyses a lesson on ratio and proportion

Module 8 Ratio and proportion 2

- Considers methods for solving problems involving ratio and proportion
- Discusses how mathematical problems and methods can be simplified or made more challenging to meet the needs of different pupils
- Highlights links between ratio and proportion and enlargement and similarity

Module 9 Using and applying mathematics

- Considers the nature of problem solving in Key Stage 3 and the implications for planning and teaching
- Discusses examples of using and applying mathematics from the *Framework for teaching mathematics: Years 7, 8 and 9*
- Considers types of questions that will engage pupils in problem solving and probe their understanding

Module 10 Effective oral and mental work

- Considers the importance of oral and mental work in all parts of mathematics lessons
- Discusses how to develop a programme of oral and mental starters

In general, the modules may be studied in any order, but if you choose to study Module 4, it would be best to study Module 3 first, since Modules 3 and 4 relate to the same theme. Similarly, study Module 5 before Module 6, and Module 7 before Module 8.

PLANNING YOUR STUDY TIME

On average, each module will take about 90 minutes of study time. For Module 9, you will need to allow about 2.5 hours.

'Doing' the modules by reading through them is not enough. You will gain much more from them if you try out and evaluate ideas in the classroom, and incorporate successful aspects into your teaching plans. If you are intending to study more than one module, make sure that you leave space between them so that you can try out and refine ideas in class.

Try to get some support or mentoring for your study, perhaps from your head of department or another experienced mathematics teacher who will act as a subject mentor. There may be points that you are unsure about and it is useful to have someone to ask or talk to. It also helps if you study the modules at the same time as another colleague so that you can discuss what you are learning as you go along.

Before you begin your study, make sure that you are familiar with:

- section 1, Guide to the Framework, of the *Framework for teaching mathematics: Years 7, 8 and 9*;
- the general layout of the yearly teaching programmes, section 3 of the Framework, and how they link to the examples in section 4, the supplement of examples.

In each module, time is allowed for you to reflect on your stage of development, to study sections of the *Framework for teaching mathematics: Years 7, 8 and 9*, and to think about and make a note of action points arising from your reflections.

Aim to build up your own learning file as you study. You can then refer back to it to gauge your progress. You can also have it by your side when you are later planning, trying and refining your teaching approaches.

RESOURCES YOU WILL NEED

Each module identifies for you the essential and desirable resources that you will require. The desirable resources can often be downloaded as PDF files from the Internet. Most are available via the mathematics publications list on the Key Stage 3 website (www.standards.dfes.gov.uk/keystage3). Detailed web references are given, where available, in each module.

In all the modules, you will need to equip yourself with:

- a personal file for inserting resource sheets and making notes as you work through the activities in the modules;
- the *Framework for teaching mathematics: Years 7, 8 and 9* (DfES 0020/2001).

In some of the modules, you will need:

- a scientific calculator;
- a pencil, ruler, compasses and plain paper.

THE VIDEO

Six of the modules make use of video excerpts. These are available on a CD-ROM called *Mathematics study modules CD Edition* which may be available from Prolog (DfES Ref 1292-2005-CDO-EN).

The lesson extracts shown in the video illustrate how mathematics teaching is developing in the schools that were filmed. The extracts are not intended as examples of 'perfect' lessons but have been chosen so that you can reflect on them as part of your professional development. When you are watching the video, you will be asked to focus on particular aspects of teaching and learning.

Sequence	Module	Duration	Title	Description
1	1	13 min	A Year 7 subtraction lesson	Catherine explores subtraction strategies with a Year 7 class
2	4	11 min	A Year 8 algebra lesson	Julie teaches her Year 8 middle set a lesson on using algebraic notation
3	5	5 min	A Year 8 geometry lesson	Bola uses dynamic geometry to develop ideas of geometrical proof with a Year 8 mixed-ability group
4	7	13 min	A Year 7 lesson on ratio and proportion	Walt teaches one of a sequence of lessons on ratio and proportion to a Year 7 top set
5	8	9 min	Year 6 pupils discussing test questions	Year 6 pupils talk about the approaches they used in questions on ratio and proportion in Key Stage 2 National Curriculum tests
6	10	8 min	Oral and mental starters	Excerpts from oral and mental starters in Year 7 and Year 8 lessons

The Key Stage 3 Strategy is grateful to all the teachers and pupils of the schools that feature in the video.

FURTHER STUDY

You may also be interested in using the DfES study guides *Teaching and learning in secondary schools*. Twenty study units exploring many aspects of the teaching repertoire, together with supporting videos, are being sent to all maintained secondary schools in England during 2004.

Key Stage 3 National Strategy: Mathematics study modules

An introduction for mentors

AIMS OF THE STUDY MODULES

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The modules are based on the two-part course, *Planning and teaching mathematics*, that has been offered to practising teachers during the early years of the Strategy.

The aims of the modules are to:

- develop teachers' understanding of important aspects of the mathematics curriculum in Key Stage 3;
- strengthen teachers' planning and teaching of mathematics in Key Stage 3.

It would be of considerable benefit to teachers studying the modules to have a mentor with whom to discuss their progress and any difficulties that they have. The mentor may be the teacher's head of department or another experienced teacher, or perhaps an advanced skills teacher, an advisory teacher or consultant. This guidance and introduction to the modules is intended for those who may be taking on that mentoring role. Separate guidance is available for teachers who are intending to study the modules.

CHOOSING STUDY MODULES

There is no need for any teacher to study all the modules unless they want to. Each is designed as a self-contained unit of work. They should choose those that most interest them and that you and they feel will be most useful. It also helps if two or three teachers study the modules at the same time so that they can discuss what they are learning and compare notes as they go along.

Descriptions of the ten modules are as follows:

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PLANNING AND SUPPORTING STUDY TIME

On average, each module will take about 90 minutes of study time. For Module 9, teachers will need to allow about 2.5 hours. You may need to stress that 'doing' the modules by reading through them won't be enough. There is much more for teachers to gain from the modules by discussing ideas with you, trying them out and evaluating them in the classroom, and incorporating successful aspects into their teaching plans. If they are intending to study more than one module, make sure that they leave space between them so that they can try out and refine ideas in class.

The most effective help you can provide is probably through a face-to-face meeting before and after each module. You will probably need to make yourself aware of the content of the selected modules. There is no need for you to mark or assess any of the tasks, but be prepared to ask questions and to uncover any difficulties. After each module, discuss with the teacher what was the impact on pupils, particularly their attitude and learning, when they tried new ideas in the classroom. Establish how it will affect their future planning. You may also be able to anticipate potential areas of difficulty in the next module and provide some preparatory help.

Before teachers begin their study, they should be familiar with:

- section 1, Guide to the Framework, of the *Framework for teaching mathematics: Years 7, 8 and 9*;
- the general layout of the yearly teaching programmes, section 3 of the Framework, and how they link to the examples in section 4, the supplement of examples.

In each module, time is allowed for teachers to reflect on their stage of development, to study sections of the *Framework for teaching mathematics: Years 7, 8 and 9*, and to think about and make a note of action points arising from their reflections. These action points are crucial. As the teacher builds up their personal file, suggest finding time periodically to review progress together, looking back at what have been the main learning points. The file should become a resource they can refer to when they are later planning, trying and refining their teaching approaches.

RESOURCES NEEDED

Each module identifies the essential and desirable resources that teachers will require. You may have in school copies of the desirable resources that could be loaned. Alternatively, the desirable resources can often be downloaded as PDF files from the Internet. Most are available via the mathematics publications list on the Key Stage 3 website (www.standards.dfes.gov.uk/keystage3). Detailed web references, where available, are given in each module.

In all the modules, teachers will need to equip themselves with:

- a personal file for inserting resource sheets and making notes as they work through the activities in the modules;
- the *Framework for teaching mathematics: Years 7, 8 and 9* (DfES 0020/2001).

In some of the modules, they will need:

- a scientific calculator;
- a pencil, ruler, compasses and plain paper.

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The lesson extracts shown in the video illustrate how mathematics teaching is developing in the schools that were filmed. The extracts are not intended as examples of ‘perfect’ lessons but have been chosen so that teachers can reflect on them as part of their professional development. When they are watching the video, they will be asked to focus on particular aspects of teaching and learning. You may find it helpful to watch some of the clips together and discuss the issues raised.

MODULE

1

Approaches to calculation

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- discusses the development of calculation strategies from Key Stage 2 to Key Stage 3 to build on pupils' knowledge, skills and understanding;
- considers teaching methods to support the development of calculation strategies, using subtraction as a focus.

CONTENT

The module is in five parts.

1 Mental calculation

2 The use of jottings in mental calculation, with special reference to subtraction

3 Written calculation

4 Teaching subtraction in Year 7

5 Summary

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- The *Framework for teaching mathematics: Years 7, 8 and 9* (DfES 0020/2001)
- The resource sheets at the end of this module:
  - 1a Teaching mental calculation
  - 1b Approaches to mental calculation
  - 1c Extending pupils' mental skills
  - 1d Using a blank number line to support subtraction
  - 1e Teaching written calculation
  - 1f Summary and further action on Module 1
- Video sequence 1, a Year 7 subtraction lesson, from the CD-ROM accompanying this module
- Section 6, the supplement of examples for Years 4, 5 and 6, from the *Framework for teaching mathematics from Reception to Year 6* (DfES ref: NNFT)
  If you don't have a copy this document, you can download section 6 from
  http://www.standards.dfes.gov.uk/primary/publications/mathematics/math_framework/teachingprogrammes/year6/55905

Desirable

- *Teaching written calculations: guidance for teachers at Key Stages 1 and 2*
  This publication (reference QCA/99/486) can be ordered at £3 per copy from QCA Publications, PO Box 99, Sudbury, Suffolk CO10 2SN (tel: 01787 884444).

STUDY TIME

Allow approximately 90 minutes.

Part 1 Mental calculation

1 This module discusses calculation methods. During your study, you will consider mental and written methods of calculation, and will watch a lesson that focuses on the teaching of subtraction. The use of the calculator is discussed in Module 2.

2 Mathematics teachers in Key Stage 3 have noticed a difference in the calculation skills of pupils transferring from Key Stage 2 in the last few years. Take a moment to consider what differences you and your colleagues have noticed in the standards of pupils' calculating skills.

Would you agree that Key Stage 3 pupils in your school:

- are more confident in calculating mentally;
- have a wider range of mental calculation strategies;
- are more willing and able to describe their calculation methods?

3 Read **Resource 1a, Teaching mental calculation**. This short article describes the mental calculation skills and opportunities to learn that pupils need to develop and experience throughout Key Stage 3.

Allocate a page of your personal file on which you can note points to discuss later with your head of department. Now think about the article you have just read. To what extent do the skills and opportunities described correspond with your experiences as a teacher of mathematics? Jot down up to three main differences for discussion with your head of department.

4 Try the calculations on **Resource 1b, Approaches to mental calculation**.

Think about the knowledge and understanding that underpins each calculation. For nearly all of the calculations, shown again below, you should expect the majority of Year 7 pupils to calculate the answers in their heads.

How do these methods differ from your methods?

$19 \times 6 = 114$	Find 20×6 then subtract 6 (mental method)
$67 \times 7 = 469$	Find 60×7 and 7×7 and add the results (may need informal jotting)
$85.5 \div 10 = 8.55$	Shift digits one place to the right (mental method which depends on multiplying and dividing by powers of 10, an important skill)
5% of 84 = 4.2	Find 10% then halve the answer (mental method)
$5.2 \times 20 = 104$	Multiply 5.2 by 10 then double (mental method)
$14.2 \div 4 = 3.55$	Halve then halve again (mental method, possibly with an intermediate note of 7.1 which is harder to halve)
$0.12 \times 0.6 = 0.072$	Multiply 12 by 6 then adjust (needs secure knowledge of relative size of numbers)
$\frac{3}{4}$ of 56 = 42	Find $\frac{1}{4}$ of 56 then multiply by 3 (mental method)

$\sqrt[3]{64} = 4$	Use knowledge of powers (i.e. know that $4^3 = 64$) or use trial and improvement

5 The *Framework for teaching mathematics: Years 7, 8 and 9* provides an overview of the progression in mental calculation skills in Key Stage 3. Study the objectives for mental calculation strategies, mental methods and rapid recall of number facts in the yearly teaching programmes for Years 5 to 9 (see the Key Stage 3 Framework section 3) and the related examples (see section 4, pages 88–103).

To what extent do these objectives correspond with your experiences as a teacher of mathematics? If there are differences, add up to three points to your notes to discuss later with your head of department.

6 Schools with Key Stage 3 pupils have already begun to ensure that pupils' mental skills are kept sharp and are developed in Key Stage 3.

Consider the three questions on **Resource 1c, Extending pupils' mental skills**. Make some notes in your personal file on what you might need to do to build on what is already in place and working well.

7 Now compare the notes you have made on Resource 1c with the suggestions below. Refine or add to your notes if you see relevant ideas.

Planning and teaching

- Include regular practice of mental skills in your teaching plans.
- Build the teaching of mental calculation strategies into your teaching plans.
- Plan for progression throughout the key stage – not just in units involving number and calculation.
- Expect pupils to use mental methods as a *first* resort – always asking themselves: 'Can I do this in my head?'
- Provide plenty of opportunities for pupils to explain, discuss and evaluate their mental calculation strategies and those of others.

Integrating mental calculation into starters and main teaching activities

- Use oral and mental starters for practising skills already taught.
- Use the main part of lesson to teach skills, explaining and discussing different methods for more complex calculations.
- Build in questions that can be answered mentally when pupils are working on different topics – for example, on averages or volume or algebraic work.
- Oral and mental starters can be used to rehearse calculation skills needed in the main part of lesson.

Ensuring suitable differentiation

- Target questions carefully to ensure the involvement of all pupils, whether in mixed-ability or setted groups.
- Use practical resources to ensure maximum participation – for example, small whiteboards, number cards or fans.
- Use resources such as number lines, number grids and place value charts to support pupils where appropriate.
- During the main part of the lesson:

- group pupils so that you can directly teach those with a particular need;
- have ready some non-routine, challenging problems to extend the most able.

- If possible, use a teaching assistant to offer targeted support to an identified group.

Part 2 The use of jottings in mental calculation, with special reference to subtraction

1 As calculations become more complex, it can be hard for pupils to hold all the intermediate steps of a calculation in their heads. At this stage, recording some or all of the steps as informal jottings becomes part of the mental strategy.

To illustrate the ideas involved, we will focus on subtraction, and in particular on the use of a blank (or empty) number line.

Read **Resource 1d, Using a blank number line to support subtraction**.

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2 In your personal file, try some examples of subtraction calculations using a blank number line approach. For example:

- use 'counting back' to work out 435 – 178;
- use 'counting down' from the larger to the smaller number to find the difference between 435 and 178;
- use 'counting up' from the smaller number to the larger number to find how many more than 178 is 435.

Which of these three methods did you prefer?

Now try working out 435 – 178 by translating it to an easier calculation.

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3 To what extent do you agree with these statements?

The blank number line (BNL):

- deepens pupils' understanding of numbers and operations;
  The BNL is a powerful way of modelling operations on numbers. It deepens pupils' understanding of the number system by helping them to focus on the characteristics of numbers (their absolute size, their relative size, their proximity to 'friendly' numbers, and so on), and not just the operation.
- helps pupils to develop and refine their mental methods;
  The BNL helps pupils to construct mental approaches based securely on their level of understanding. It can also help them to refine their existing mental methods and extend their methods to numbers that are more difficult.
- makes a useful diagrammatic record of calculation strategies;
  Pupils can use the BNL to help explain to themselves and others how they have tackled a calculation. It stimulates explanation and discussion of calculation strategies.
- can be extended to calculations involving negative numbers and decimals.
  As such, the BNL is a tool that should remain familiar to all pupils and not be seen merely as a support tool for those with poor calculation skills.

Part 3 Written calculation

1 Read **Resource 1e, Teaching written calculations.**

2 The *Framework for teaching mathematics from Reception to Year 6*, which should be readily available in all secondary schools, provides valuable guidance on the progression to written calculations. Look at pages 48–51 of section 6, the supplement of examples for Years 4, 5 and 6.

Find the objective: 'Develop and refine written methods for subtraction, building on mental methods'. Consider progression across Years 4, 5 and 6 by looking at the headings in each column: informal written methods; standard written methods; extend to decimals. Track the progression through the years by scanning across the columns.

Now look at pages 66–69 of section 6, which illustrate progression in multiplication and division.

Part 4 Teaching subtraction in Year 7

1 Consider this subtraction.

$$\begin{array}{r} 1000 \\ \underline{-\ \ 99} \end{array}$$

- How would you expect Year 7 pupils in you school to tackle this question?
- What would you do next with pupils who attempted the standard method of decomposition but got it wrong?

In your personal file, jot down your thoughts on these two issues before continuing.

2 Watch **Video sequence 1, a Year 7 subtraction lesson**. The lesson is taught by Catherine. All pupils in the class achieved either level 3 or level 4 in the Key Stage 2 National Curriculum tests. In a previous lesson, Catherine gave pupils a short test to assess their competence in using written methods of subtraction for whole numbers.

The video sequence lasts about 13 minutes.

When you have finished watching, spend a few minutes considering how Catherine's approach to tackling her pupils' difficulties with subtraction compares with what you would have done.

3 Read the notes below. These give some background information on the teaching and learning activities that you have seen in the video.

- Many pupils believe that because a subtraction is presented vertically it must be tackled using decomposition. For this reason, questions such as 1000 – 99 tend to be presented horizontally in the Key Stage 2 and Key Stage 3 National Curriculum tests.
- Pupils who struggle with decomposition might be able to tackle 1000 – 99 in their heads without any writing.
- A blank number line provides useful support for pupils' informal methods.
- The blank number line is only one of several possible models that can help to reinforce pupils' understanding of subtraction.

- Giving pupils opportunities to work in pairs to discuss calculation strategies can deepen their understanding. Observing this kind of discussion gives teachers valuable evidence of pupils' thinking.
- The teaching of the decomposition method of subtraction needs to stress the place value of digits. For example, in changing:

  85 to $^{7}\cancel{8}\,{}^{1}5$,

  say 'eighty and five becomes seventy and fifteen' rather than 'eight becomes seven with a little one beside the five'.

Part 5 Summary

1 Some important principles in the teaching of calculation strategies are:

- Allow pupils to show what they know and can do.
  Establish pupils' competences before trying to move them forward. Effective assessment ensures that pupils are allowed to show what they know and can do. Don't rely solely on narrow tests of competency in particular written methods.
- Plan to maintain and develop mental calculation strategies.
  Maintain and extend pupils' repertoire of mental approaches throughout Key Stage 3 through regular planned practice and focused teaching.
- Value and build on informal jottings.
  Informal jottings (for example, using a blank number line) are legitimate support for harder mental calculations. They should be valued and used as the basis of discussions, not hidden on scraps of paper or in the backs of exercise books!
- If pupils consistently struggle with a calculation method, backtrack to an earlier stage to establish firm understanding.
  For pupils who are not secure with formal written approaches, it is important to go back to building confidence with informal methods that reflect their level of understanding.
- Teach pupils to discuss their own and others' strategies.
  Encourage pupils to talk through their approaches, paying attention to the necessary vocabulary. This allows them to exchange and evaluate different approaches. The discussion also provides teachers with valuable insights into pupils' understanding.
- Emphasise 'doing it in your head if possible' and 'checking your answer'.
  Remind pupils constantly to ask: 'Can I do this in my head?', 'How can I check my answer?' and 'Does my answer make sense?'

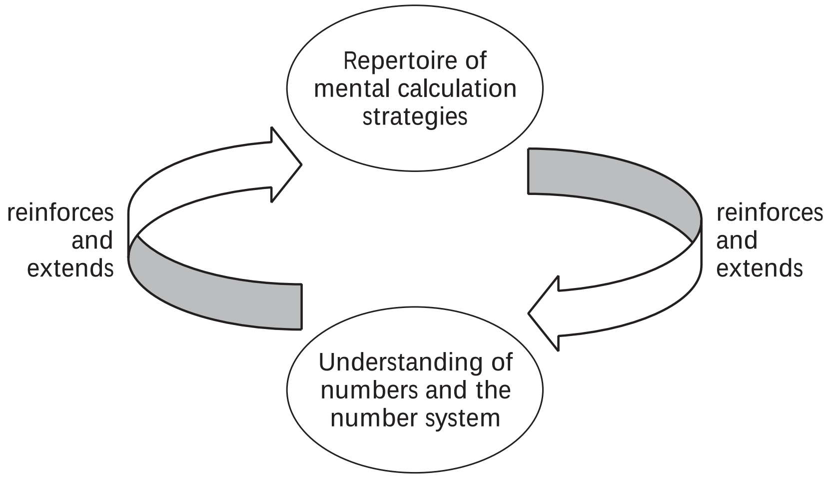
2 Look back over the notes you have made during this module. Have you identified the most important things that you may need to consider and adopt in your planning and teaching of calculations?

Use **Resource 1f, Summary and further action on Module 1**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and the main points to discuss with your head of department.

3 You will find useful guidance on teaching written calculations in *Teaching written calculations: guidance for teachers at Key Stages 1 and 2* (see the first page of this module for ordering information).

Resource 1a Teaching mental calculation

The ability to calculate mentally lies at the heart of numeracy. The diagram shows the relationship between two elements of pupils' learning of mathematics: the development of calculation strategies (particularly mental strategies) and their understanding of numbers and the number system. Each element supports and depends on the other, so teachers need to plan carefully to address both elements in tandem in their teaching.



A number of skills are involved that pupils need to practise and extend throughout Key Stage 3. These skills are:

- recognising that mental approaches should be considered as a first resort;
  i.e. remembering to ask: 'Can I do this in my head, without any writing?' and, if not: 'Can I do it in my head with some jottings to help?'
- remembering number facts and recalling them without hesitation;
  e.g. recalling facts such as multiplication and division facts, the percentage and decimal equivalents for common fractions, squares and cubes, …
- using known facts to figure out new facts;
  e.g. knowing that if $3 \times 6 = 18$ then $3 \times 0.6 = 1.8$ and $300 \times 60 = 18\,000$.
- drawing on a repertoire of mental strategies to work out calculations, with some thinking time, and being able to explain methods;
  i.e. being able to call on a range of strategies and choose strategies appropriate for the calculation.
- understanding and using relationships between the four operations to find answers and check results;
  e.g. knowing that if $20 \times 36 = 720$ then $720 \div 20 = 36$.
- approximating calculations to judge whether an answer is reasonable;
  e.g. knowing that $490 \div 24$ will be roughly the same as $500 \div 25$, i.e. approximately 20, and use this to check that the answer is sensible.
- solving problems, including identifying what operations to use and the steps to take.

Teachers need to make sure that pupils have plenty of opportunities to:

Decimal partitioning

3.478
$= 3 + 0.4 + 0.07 + 0.008$
$= 3 + \frac{4}{10} + \frac{7}{100} + \frac{8}{1000}$

- strengthen their understanding of the number system, including place value, decimal partitioning, and so on;
- describe and explain their mental calculation methods;

- appreciate that different methods will work for a particular calculation, and to discuss which are the most efficient;
- deepen their awareness of how the laws of arithmetic apply to mental calculations, and apply these laws in the context of algebra;
- continue to practise the mental calculation skills they have already developed in Key Stage 2;
- extend their mental skills to cope with more complex calculations, including those involving decimals, fractions, percentages, powers, conversions of measurements, and so on.

Laws of arithmetic

commutative law

$8 + 47 = 47 + 8$
$15 \times 36 = 36 \times 15$

associative law

$13 + (7 + 8) = (13 + 7) + 8$
$8 \times (5 \times 9) = (8 \times 5) \times 9$

distributive law

$(20 \pm 3) \times 7$
$= (20 \times 7) \pm (3 \times 7)$

$(60 \pm 9) \div 3$
$= (60 \div 3) \pm (9 \div 3)$

Resource 1b Approaches to mental calculation

Do the following calculations mentally, then make a note of your method.

19×6

67×7

$85.5 \div 10$

5% of 84

5.2×20

$14.2 \div 4$

0.12×0.6

¾ of 56

$\sqrt[3]{64}$

Which calculations would you *expect* pupils to attempt mentally?

What kinds of jotting help to support pupils' mental methods?

Resource 1c Extending pupils' mental skills

Focus on each of these three questions in turn. For each question, jot down three things that you could do **to build on and extend pupils' mental skills**.

How could you strengthen your planning or teaching of mental skills in mathematics lessons? • • •

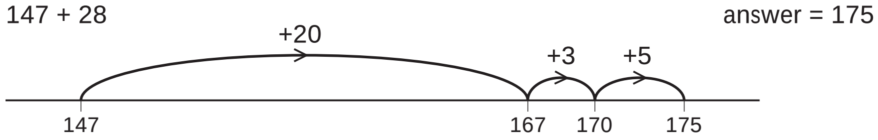
How could you further the ways in which you integrate mental calculation into both the oral and mental starter and the main teaching activities? • • •

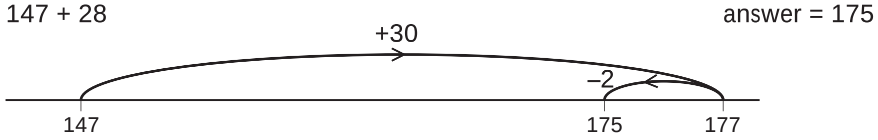
How could you strengthen the ways in which you provide for differentiation? • • •

Resource 1d Using a blank number line to support subtraction

Recent research has shown that the blank (or empty) number line can be a powerful model for building understanding of and developing calculation strategies.

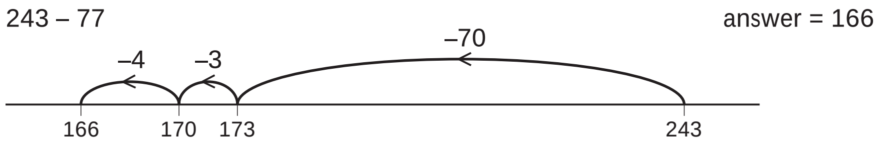
First, here are two examples of the blank number line representing addition.

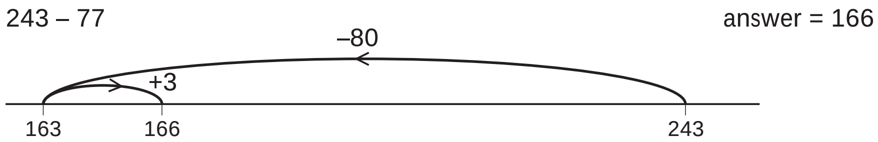




Both of these model common mental approaches. In each case, we start by marking the first number (or, if using a more sophisticated approach, the smaller number). We then count *on from* that number *by* the amount we are adding on. In the second example, the fact that 28 is close to a decade number (30) prompts the compensation approach illustrated. Note that the answer appears as a position on the number line.

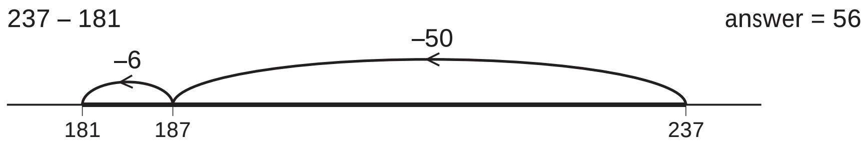
One model of subtraction is the opposite of 'adding on' (i.e. subtraction is the inverse of addition). To illustrate this on the number line we count *back from* the first number *by* the amount we are subtracting. Here are two possible representations. Note again that the answer appears as a position on the number line.

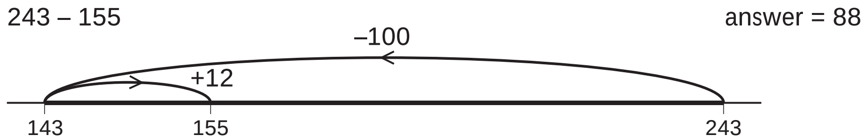




An alternative and powerful model for representing subtraction is to see it as the difference between two numbers. Pupils need to be familiar with this view. On the number line, the idea is represented as the length of the line segment between the two numbers (the thicker lines in the examples below). This length can be calculated in different ways. To start, the two numbers whose difference is sought are marked on the line. This action helps focus on the characteristics of the two numbers which, in turn, influences the approach taken.

In this first example, the difference is calculated by counting *down* from the larger number *to* the smaller.

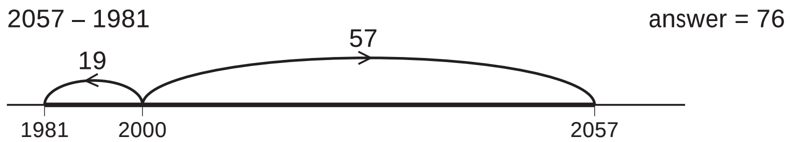




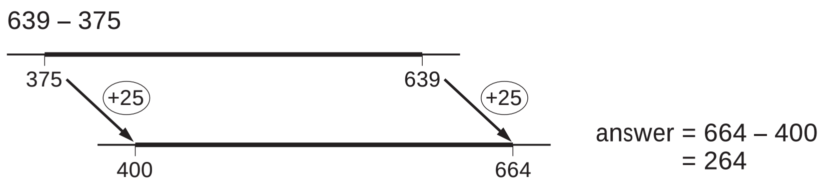
The choice of steps will depend upon how the pupil sees the numbers. Note that the answer is now the gap, not a position on the line.

An alternative approach, clearly equivalent, is to count *up* from the smaller *to* the larger number. This is analogous to a shop assistant (with no electronic till) counting out change. Again, the actual steps will depend upon the numbers involved and pupil preferences.

For some pairs of numbers a useful model is 'jumping out' from an obvious 'bridge point'.



Finally, this picture of difference leads to the idea of translating the pair of numbers up or down the number line without changing the gap between them. In this way, it is possible to translate one subtraction to another with the same answer but involving friendlier numbers. For example, 639 – 375 is equivalent to 664 – 400, which is an easy mental calculation.



Familiarity with this dynamic image can help pupils overcome difficulties they encounter when using compensation methods mentally. For example, using the calculation above, a pupil spotting that 375 is near 400 might begin by subtracting the 400 from 639 to get 239. She now has to compensate for the additional 25 subtracted but may be unsure whether to add it to or subtract it from 239. In order to 'keep the gap the same' she clearly needs to add it.

Resource 1e Teaching written calculation

Nearly all pupils will have been introduced to formal written calculation methods by the end of Key Stage 2. A significant minority of pupils may not have a secure enough understanding of number to cope with some standard methods. The compactness of traditional standard methods, particularly working with digits without referring to their values, prevents some pupils from understanding why a method works. This is especially true of subtraction and long division.

There is evidence from Key Stage 2 tests that pupils can make errors because they try to use a standard method (such as decomposition for subtraction) when they are not secure with it. These pupils would be more successful if they used an informal method which they fully understand and with which they are more confident.

The aim is that pupils choose appropriate calculation strategies that they use accurately, efficiently and with understanding. To achieve this, all teachers of mathematics need to understand the progression from informal mental methods of calculation to efficient, compact written methods for addition, subtraction, multiplication and division. The QCA booklet *Teaching written calculations: guidance for teachers at Key Stages 1 and 2* illustrates this progression. It was written originally for primary teachers but has useful guidance for teaching Key Stage 3 pupils who are not yet consistently competent in using such methods. All secondary schools were given a copy of the booklet in the summer of 2000.

The progression towards standard written methods can be summarised as:

- establishing mental methods, based on good understanding of place value;
- informal recording of mental methods, becoming more structured;
- more formal expanded written methods for addition and subtraction, leading to more compact standard methods;
- informal methods for multiplication and division, built on an understanding of mental strategies;
- standard methods used efficiently and accurately, and with understanding, with increasingly more complex calculations.

Informal recording to support or explain mental calculations

For example:

43

$$\begin{array}{ccccc} 40 & + & 3 & & \\ \downarrow & & \downarrow \times 6 & & \\ 240 & + & 18 & = & 258 \end{array}$$

×	40	3
6	240	18

240 + 18 = 258

Some factors that Key Stage 3 teachers need to consider when they are planning and teaching are as follows.

- Pupils should continue to be expected to use mental methods as a first resort, not only in mathematics lessons but also in other subjects.
- Formal written methods for addition and subtraction should be established by Year 6 and used in Key Stage 3 with an increasing range of whole numbers and decimals.
- Multiplication and division methods will need to be developed further. Some pupils may still use informal written methods, such as:
  - a 'grid' method for multiplying two- or three-digit numbers, which can be consolidated and built on in Key Stage 3 and related to work in algebra;
  - a 'chunking' method for division, which will also need to be built on and extended in Key Stage 3.
- With written methods, pupils should be able to use approximate values and mental calculations to estimate the size of the answer for checking purposes.

Resource 1f Summary and further action on Module 1

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching of calculation.

List two or three key points that you have learned.

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List two or three points to follow up in further study.

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List two or three modifications that you will make to your planning or teaching of calculation.

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List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

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MODULE

2

Using calculators

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- discusses when it is and when it is not appropriate for pupils to use a calculator;
- considers the skills that pupils need to use a calculator effectively and efficiently;
- discusses the progression of calculator skills and the implications for teaching.

CONTENT

The module is in four parts.

1 The role of the calculator

2 Mental, written or calculator?

3 Progression in calculator skills

4 Summary

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- A scientific calculator
- The *Framework for teaching mathematics: Years 7, 8 and 9*
- The resource sheets at the end of this module:
  - 2a How do you currently use calculators?
  - 2b Using a calculator
  - 2c Learning with a calculator
  - 2d Mental, written or calculator?
  - 2e Progression in calculator skills
  - 2f Teaching points for Year 7
  - 2g Summary and further action on Module 2

Optional

- 16 coloured counters, eight of one colour and eight of another (or use two different kinds of coins)

STUDY TIME

Allow approximately 90 minutes.

Part 1 The role of the calculator

1 This module is about the use of the calculator in Key Stage 3 mathematics lessons. During it, you will be considering the skills that pupils should be developing in Years 7, 8 and 9 and the kinds of activities that will help them to develop those skills.

2 Consider the three questions on **Resource 2a, How do you currently use calculators?** Use the resource sheet to make notes on the current position in your school.

Could the teaching of calculator skills in your school be more systematic? If so, working through this module should help you to achieve this, particularly in Year 7.

3 Read and fill in the table on **Resource 2b, Using a calculator**.

To what extent do your answers to the four statements correspond with your perspective on current practice in Resource 2a? If there are differences, jot down the three main ones to discuss later with your head of department.

4 The two main ways in which calculators can play a role in mathematics lessons are as a teaching or learning aid, and as a calculating aid. **Resource 2c, Learning with a calculator**, has some examples of calculator activities that can support learning. Try the first two activities, then consider and make notes on the accompanying questions.

5 The first two activities on Resource 2c are puzzles or investigations that pupils can work on independently or in pairs. Other activities are in the form of a game for two or more players. The third activity on Resource 2c, reproduced below, provides an example.

Four in a row: a game for two players

Each player needs about 8 counters in their own colour.

506	1426	217	837	1136	3266
4757	1809	1242	3082	341	112
77	496	3752	432	176	2201
1917	736	737	189	2576	1072
616	322	896	781	3976	497
469	1512	1736	2077	392	297

Take turns.

Choose a number from the playing board and two numbers from this list.

7 11 16 27 31 46 56 67 71

The player finds the product of the two numbers using a calculator. If it equals the chosen number, the player covers the number with a counter.

The winner is the first player to get four of their counters in a straight line.

Activities like those on Resource 2c can be presented to individual pupils or groups of pupils on a worksheet or to the whole class on an overhead projector transparency. They are just three of many different examples that can help to develop or practise calculation skills and ideas of number.

A further advantage is that where the calculator is used to check the accuracy of calculations, pupils can practise aspects of calculation without disturbing the teacher, freeing the teacher to observe or to provide direct teaching to selected small groups.

Part 2 Mental, written or calculator?

1 Work through the questions on **Resource 2d, Mental, written or calculator?**, deciding what kind of calculation to do.

You could use an exercise with questions similar to those on Resource 2d as an oral and mental starter to a mathematics lesson. This would reinforce for pupils the importance of selecting the most appropriate calculation method for a particular question.

In your personal file, plan an exercise along the same lines that you could give to a class of Key Stage 3 pupils in, say, the next month.

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2 Compare the recommendations below with your decisions on the questions on Resource 2d.

1 Mental
A simple test of divisibility confirms that the answer will be a whole number of pounds.

2 Mental
The convenience of the numbers here means the calculations are easily done mentally. Confirm the answers by checking that they total 420.

3 Calculator
A clear candidate for the calculator (although formal or informal written methods are appropriate). How can the calculator be used efficiently?

4 Mental (for some), calculator or written (for others)
Remember to estimate first (perhaps using $125 \times 8 = 1000$*).*

5 Calculator
A calculator can quickly help find the total number of minutes, but how best to convert to hours and minutes will need to be taught.

6 Mental
This should be seen as a mental calculation ('counting on': 1 h 15 min + 5 h 20 min, assuming both times are in the same time zone).

7 Mental
The context demands an answer to the nearest whole number above so a mental method is appropriate. If $387 \div 51$ *is attempted using a calculator, the result will need to be interpreted.*

8 Calculator, written or mental
Most are likely to prefer a calculator or written approach. Noting the equivalence to $31.2 \div 12$ *would allow some to tackle it mentally.*

Key Stage 3 pupils should not be over-reliant on calculators. They should be able to recognise when it is more appropriate to use mental methods, with or without jottings, and they should be able to use written methods accurately and efficiently. Make some notes in your personal file on what you might need to do to help pupils to do this.

Part 3 Progression in calculator skills

1 Read **Resource 2e, Progression in calculator skills**.

2 The Key Stage 2 National Curriculum tests have shown that a significant number of pupils entering Year 7 may not have developed the technical skills necessary to use a calculator efficiently, although the position is improving slowly.

One of the factors that not all pupils may appreciate by the end of Key Stage 2 is the difference between the 'answer' as displayed in the calculator display, and the 'answer' that should be recorded.

Work through the questions on **Resource 2f, Teaching points for Year 7**.

3 Compare your answers to the questions on **Resource 2f** with those below.

	Display	Written answer	Teaching points
1	15.6	£15.60	Note the need to write the zero for 60p in the written answer …
2	0.8	£0.80 or 80p	… and the zero for 80p.
3	9.14 or 914	9.14 m or 914 cm	The units must be the same. Enter 8.47 + 0.67 for an answer in m, or 847 + 67 for an answer in cm.
4	0.09 or 9	9 cm	Enter 1.03 – 0.94 for an answer in m, or 103 – 94 for an answer in cm.
5	4.95 or 495	£4.95	The calculation should be done mentally. If a calculator is used, enter 8p as 0.08, or £5.03 as 503.
6	1237 or 1.237	1237 g or 1.237 kg	Enter 1500 – 263 by changing both to g, or 1.5 – 0.263 by changing both to kg.
7	0.185	18.5p or 19p to nearest p	Discuss what to do about digits beyond the second decimal place …
8	2.266666	£2.27 to nearest p	… and how to round to two decimal places.

In your personal file, plan some questions to ask Year 7 pupils that would allow you to draw attention to similar teaching points.

4 Look at the objectives identified under 'Calculator methods' and 'Checking results' in the teaching programmes for Years 7, 8 and 9, Framework section 3, pages 6–13. Look at the corresponding examples in the supplement, Framework section 4, pages 108–111.

Think about how you could make the teaching of calculator skills more explicit in your planning and teaching of Key Stage 3 lessons. Jot down your ideas in your personal file.

5 Key Stage 3 pupils will also benefit from learning how to use a graphical calculator – this should not be restricted to the most able pupils. There are several examples of activities in different contexts, including solving problems, in the Framework supplement of examples. Examples include:

- using a large screen to follow the steps of a calculation and to explore patterns in calculations and sequences;
- beginning to explore straight-line graphs and the relationship between the values of x and y;
- solving problems involving coordinates and shapes.

Look for examples of the use of graphical calculators by browsing through the supplement of examples, Framework section 4. Focus in particular on number and algebra. In your personal file, make a note of any example that would be useful to explore with the classes that you teach.

Part 4 Summary

1 Some important principles in the use of calculators are:

- Pupils need regular opportunities to practise and extend their mental skills, since the ability to estimate the result of a calculation by approximating and calculating mentally is key to the successful use of a calculator.
- Calculator skills need to be taught systematically. Pupils will not learn these technical skills merely by 'being allowed' to use calculators.
- Pupils should be encouraged to use the most appropriate method for a calculation and should always ask themselves first: 'Can I do this in my head?'
- Skilful use of a calculator depends on:
  - a good understanding of the number system, including place value and rounding;
  - a good knowledge of units of measurement, and the ability to convert one unit to another mentally.
- Calculators are more than calculating aids. They can also support the teaching and learning of different aspects of mathematics.

2 Look back over the notes you have made during this module. Have you identified the most important things that you may need to consider and adopt when you are planning and teaching?

Use **Resource 2g, Summary and further action on Module 2**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and the main points to discuss with your head of department.

Resource 2a How do you currently use calculators?

How often do you *expect* pupils to use a calculator (as opposed to 'letting' pupils use a calculator)?

How do you *teach* calculator skills?

How do you ensure that pupils use an appropriate method of calculation (mental, written or calculator)?

Resource 2b Using a calculator

Do you agree or disagree with the statements below?

	Agree	Disagree
A calculator can be used as a teaching tool to develop understanding of concepts as well as being a calculation tool.		
Calculator skills need to be taught systematically.		
Poor calculator skills currently hinder some pupils.		
Pupils should be taught to recognise when it is appropriate to use a calculator and when it is more appropriate to use a mental or written method for a calculation.		
There should be times when pupils are asked to put their calculators away and to work without them.		

To what extent do your answers to the four statements correspond with your perspective on current practice in Resource 2a?

If there are differences, jot down the three main ones to discuss later with your head of department.

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Resource 2c Learning with a calculator

1 MISSING OPERATIONS

Each circle represents a missing operation.

Find out what it is.

1 $(37 \bigcirc 21) \bigcirc 223 = 1000$

2 $(756 \bigcirc 18) \bigcirc 29 = 1218$

3 $27 \bigcirc (36 \bigcirc 18) = 675$

4 $31 \bigcirc (87 \bigcirc 19) = 2108$

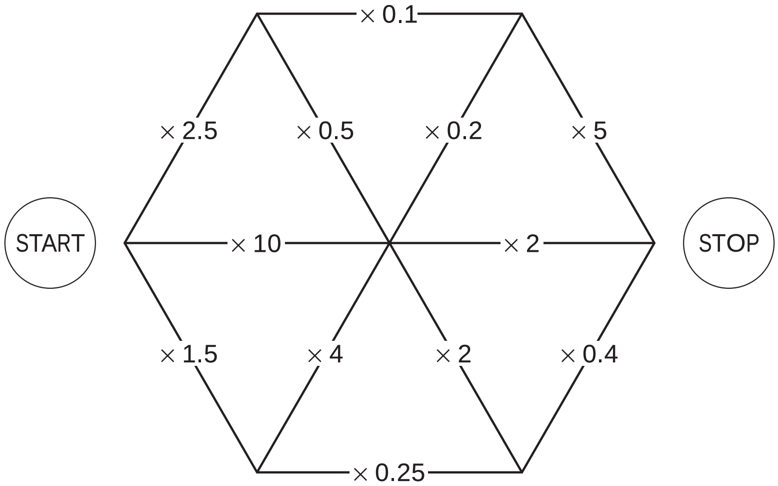
When you have solved these problems, consider these questions. Use the space below to make notes.

1. Could problems like this be used with Key Stage 3 pupils?
2. What technical skills in using calculators would they need?
3. What mathematical skills and understanding could they develop through working on the problems?
4. Could you present these or similar problems to the whole class on an overhead projector?
5. How could you adapt the problems to make them easier for less able pupils? Or more challenging for more able pupils?

Notes:

2 MAZE

Try this puzzle. Start with 1 in your calculator display.



Choose a route from START to STOP.

You may go along each line only once.

Multiply the number in your display by the number on the line.

The aim is to finish with 5 in your display.

When you have solved this puzzle, consider these questions. Use the space below to make notes.

1. Could the puzzle be used with Key Stage 3 pupils?
2. What technical skills in using calculators would they need?
3. What mathematical skills and understanding could they develop through working on the puzzle?
4. Could you present the puzzle to the whole class on an overhead projector?
5. How could you adapt the puzzle so that it could be played as a game by two players?
6. How could you adapt the puzzle to make it easier for less able pupils? Or more challenging for more able pupils?

Notes:

3 FOUR IN A ROW

This is an example of a game for two players. If you wish, you could find a partner and try playing the game.

Each player needs about 8 counters in their own colour (or you could use two different kinds of coins).

506	1426	217	837	1136	3266
4757	1809	1242	3082	341	112
77	496	3752	432	176	2201
1917	736	737	189	2576	1072
616	322	896	781	3976	497
469	1512	1736	2077	392	297

Rules

Take turns.

Choose a number from the playing board and point it out. Then choose two numbers from this list.

7 11 16 27 31 46 56 67 71

Find the product of the two numbers using a calculator. If it equals your chosen number, cover the number with a counter.

The winner is the first player to get four of their counters in a straight line in any direction, horizontal, vertical or diagonal.

If you have a chance to play *Four in a row*, consider these questions. Use the space below to make notes.

1 What mathematical skills and understanding could Key Stage 3 pupils develop through playing this game?

2 Could you present the game to the whole class on an overhead projector?

3 How could you adapt the puzzle to give pupils experience of working with different kinds of numbers (e.g. decimals with one or two places, positive and negative numbers, fractions)?

Notes:

Resource 2d Mental, written or calculator?

Work through these questions. For each question decide which method of calculation a Key Stage 3 pupil might use, and which method a teacher like yourself would use – mental method, written method, or calculator method?

		Method a pupil might use	Method a teacher might use
1	Three friends shared £411. How much did each friend get?		
2	John owns 420 CDs. The ratio of his classical to pop CDs is 3 : 4. How many of each type of CD does John own?		
3	Rashida travelled 532 miles last month for work. She is paid 35p a mile for the first 250 miles and 20p a mile for the remainder. How much will she be paid?		
4	$6250 \div 125 =$		
5	A turkey needs to be cooked for 40 minutes per kilogram, plus 20 minutes. For how long would you cook a turkey that weighs 5.2 kg?		
6	A ferry leaves Hull at 22:45 and arrives in Rotterdam at 05:20 the next day. How long did the journey take?		
7	372 pupils and 15 adults are going on an outing. How many 51-seater coaches will they need?		
8	The mass of an object is 0.312 kg. Its volume is 0.12 m^3. Density is mass divided by volume. What is the density of the object?		

Resource 2e Progression in calculator skills

To ensure good continuity and progression in pupils' learning, Key Stage 3 teachers need to be aware of the progression in calculator skills from Year 5 through to Key Stage 3.

The calculator skills that most pupils in Years 5 and 6 are expected to develop are as follows. Pupils should learn to:

- use a calculator to perform a one-step problem and interpret the result;
  Most pupils will have little difficulty with entering a one-step calculation such as 4.5×27. However, when they are solving word problems, when it is not always obvious which values and what operations to use, they may misinterpret the question and enter the wrong calculation.
- key in and interpret money and measurement calculations;
  Interpreting the results of money calculations often causes difficulties; for example, recognising whether the answer is in pence or pounds and interpreting the result of a division calculation.
- recognise rounding errors – for example, recognise 2.9999999 as 3;
  Pupils need to make sense of problems and realise when answers are likely to have been rounded.
- use division to enter a fraction such as $\frac{3}{8}$, recognising the display of 0.375 as the decimal equivalent;
- recognise recurring decimals, such as 0.3333333, and know that this is equivalent to $\frac{1}{3}$;
  Pupils need to recognise that not all digits may recur in a decimal, as in $1 \div 6 = 0.16666666$.
- recognise negative numbers and use the sign-change key if appropriate;
  Pupils may miss the minus sign that indicates a negative number, which is usually on the extreme left of the display.
- carry out calculations with more than one step, such as $8 \times (37 + 58)$, or $\frac{3}{8}$ of 980;
  Pupils need to be familiar with the order of operations so that they select the correct sequence of operations in calculations involving more than one step.
- clear the display before starting a calculation;
  Pupils are less likely to make errors if they clear the display before starting a new calculation.
- correct a wrong entry by using the CLEAR ENTRY key;
  Most Key Stage 2 pupils will clear the display and repeat the calculation if they think that they made an error. They also need to learn how and when to use the CE key.
- have a feel for the size of an answer and check it appropriately, for example, by carrying out the inverse operation.
  This is the most important skill – errors in entering values often lead to nonsensical answers!

These skills need to be consolidated, reinforced and built upon throughout Key Stage 3. In summary, by the end of Key Stage 3 most pupils should know:

- the order in which to use the keys for calculations involving more than one step;
- how to carry out calculations involving percentages without using the percentage key;

- how to enter numbers and interpret the display when the numbers represent money, metric measurements, units of time or fractions;
- how to carry out calculations involving mixed units of time (e.g. hours and minutes, or minutes and seconds);
- when and how to use facilities such as the memory, brackets, the square-root and cube-root keys, the sign-change key, the fraction key, the constant facility;
- how to select from the display the number of figures appropriate to the context of a calculation.

Resource 2f Teaching points for Year 7

Use your calculator to answer the questions in the table below. Complete the columns headed 'Display' and 'Written answer'. Leave the column headed 'Teaching points' blank at this stage.

	Question	Display	Written answer	Teaching points
1	£6.25 + £9.35 =			
2	£3.52 – £2.72 =			
3	8.47 m + 67 cm =			
4	1.03 m – 94 cm =			
5	£5.03 – 8p =			
6	1.5 kg – 263 g			
7	50 pens cost £9.25. What does 1 pen cost?			
8	Washing machine liquid costs £3.40 for 1.5 litres. What is the cost of 1 litre?			

Now go back and complete the column headed 'Teaching points'.

Jot down what points you could draw out by asking Year 7 pupils to do a few examples of this kind. For example, for the first question, you could ask pupils to note the need to write the zero for 60p. You could also remind them to include the £ sign in their answer, but not a p for pence.

Resource 2g Summary and further action on Module 2

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching.

List two or three key points that you have learned.

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-
-

List two or three points to follow up in further study.

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-
-

List two or three modifications that you will make to your planning or teaching.

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List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

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-
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MODULE

3

Thinking about algebra

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- considers some stimulating activities for teaching an aspect of algebra, the simplification of algebraic expressions;
- discusses how the activities can also help to develop algebraic reasoning;
- considers how the activities might be incorporated in mathematics lessons.

CONTENT

The module is in five parts.

1 Algebra in Key Stage 3

2 Practising collecting like terms

3 Applying algebraic reasoning

4 Approaches to algebra in the classroom

5 Summary

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- The *Framework for teaching mathematics: Years 7, 8 and 9*
- Scissors
- The resource sheets at the end of this module:

  3a An algebra loop card game

  3b An algebraic magic square

  3c Addition squares

  3d Reflections on addition squares

  3e More addition squares

  3f Approaches to algebra in the classroom

  3g Summary and further action on Module 3

Desirable

- *Framework for teaching mathematics from Reception to Year 6* (the Primary Framework), pages 9 and 10 of section 1, the introduction (pages 9 and 10, Laying the foundations for algebra, can be downloaded from http://www.standards.dfes.gov.uk/primary/publications/mathematics/math_framework/introduction/938949
- *Interacting with mathematics in Key Stage 3: Constructing and solving linear equations* http://www.standards.dfes.gov.uk/keystage3/respub/ma_interlin
- *Teaching and learning algebra pre-19*, a joint report from the Royal Society and the Joint Mathematical Council www.royalsoc.ac.uk/document.asp?id=1910

STUDY TIME

Allow approximately 90 minutes.

Part 1 Algebra in Key Stage 3

1 This module considers one of the learning objectives from the Year 8 teaching programme for algebra: that pupils should be taught to simplify or transform linear expressions by collecting like terms. The module describes some activities that address this learning objective. It then goes on to consider the place of these and similar activities in mathematics lessons.

2 Algebra in Years 7 to 9 includes equations, formulae and identities, and sequences, functions and graphs. Read pages 14–15 of the Guide to the Framework, section 1, of the *Framework for teaching mathematics: Years 7, 8 and 9*. These pages summarise the Key Stage 3 Strategy's approach to the teaching of algebra.

The Framework makes clear that algebra in Key Stage 3 involves:

- developing pupils' understanding that algebra is a way of generalising either from arithmetic, or from particular cases or from patterns and sequences;
- providing regular opportunities to construct algebraic expressions and formulae and to transform one expression into another – for example, by collecting like terms, taking out common factors, working with inverses or solving linear equations;
- using opportunities to:
  - represent a problem and its solution in tabular, graphical or symbolic form, using a graphical calculator or a spreadsheet where appropriate;
  - relate solutions to the context of the problem;
- developing algebraic reasoning, including an appreciation that while a number pattern may suggest a general result, a proof is derived from the structure of the situation being considered.

3 Algebra is not taught formally in Key Stage 2. If you have access to it or can download the relevant pages, read pages 9 and 10 of the Introduction, section 1 in the Primary Framework. These pages summarise how the foundations for algebra are laid in Key Stage 2 through a variety of relevant experiences, although the word 'algebra' is not used to describe them.

Part 2 Practising collecting like terms

1 Spend a few minutes jotting down in your personal file the activities and contexts that you currently use to teach pupils to simplify or transform linear expressions by collecting like terms.

2 Copy, cut out and shuffle up the cards on **Resource 3a, An algebra loop card game**. Spread out the cards on a flat surface so that you can see them all. Take one of the cards at random and place it to the left of you. Find the answer to the question on this card and place it to the right of the first card to form a line. Carry on until you have used all 18 cards.

This is a self-checking activity in that the last card on the right of the line should link back to the first card on the left.

The activity is one that small groups of pupils can work on collaboratively. Alternatively, a similar set of cards can be distributed around a whole class to play an 'I have … What is …' loop card or 'follow me' game. A pupil starts the game by reading their algebraic expression and their question. Other pupils follow until all cards have been called and the 'loop' has been completed. As pupils play the game, you can write each new expression on the board for everyone to see.

Take a few moments to think about these questions.

- What is the purpose of this kind of activity?
- What could the advantages be of using an activity like this at the start of a lesson?
- How could the activity be organised or adapted it to make it suitable for pupils at different levels of attainment?

Part 3 Applying algebraic reasoning

1 The activity in Part 2 of this module provides mental practice in collecting like terms. Every pupil is involved as they think about each question to see whether the answer matches the expression on their card. The complexity of the expressions can be varied and, where pupils play the game in groups, different sets of cards can be given to different groups.

Other activities offer further mental practice but also require reasoning to solve algebraic problems. An example of this kind of activity is one involving magic squares. In a magic square, a set of numbers is arranged in a square grid so that the numbers in each row, column and the two main diagonals have the same total.

In your personal file, quickly arrange the numbers 1 to 9 to form a 3 by 3 magic square. This problem helps pupils to practise adding three single-digit numbers mentally but, in addition, requires some reasoning in order to arrive at a solution.

A teacher who wants pupils who have solved this problem to describe their methods and reasoning is likely to prompt them with questions such as:

- Where did you start?
- What did you do next? Why?
- How many different solutions are there?

Through discussion, the teacher will help pupils to appreciate that although solutions may look different, they are all reflections and rotations of each other. There is only one solution represented in different ways. The central number is the middle number of the set of numbers 1 to 9 (and is also the mean of the set).

2 The same conclusion can be reached by algebraic reasoning.

Copy, cut out and shuffle up the nine cards, each with an algebraic expression, on **Resource 3b, An algebraic magic square.** Arrange the cards to form a magic square.

When you have completed the square to your satisfaction, consider these questions.

- How did you start? How did you continue?
- Could you have started and continued in a different way?
- How did you check your solution?
- How many different solutions can you find?
- To what extent did your earlier consideration of a magic square with the numbers 1 to 9 help you to solve the problem with algebraic expressions?

Some points to remember in relation to the algebraic magic square activity are these.

- The problem demands logical reasoning and other problem-solving skills.
- The solution can be checked by determining whether the sum of the three algebraic expressions in each line is the same. It can also be checked by substituting particular values for a and b (for example, $a = 1$, $b = 0$, or $a = 0$, $b = 1$).
- There is one solution. Reflections and rotations produce other possible arrangements of this solution.

$a + 4b$	$8a + 3b$	$3a + 8b$
$6a + 9b$	$4a + 5b$	$2a + b$
$5a + 2b$	$7b$	$7a + 6b$

- Substituting different values for a and b produces an infinite number of different magic squares, all with the same structure, rules and internal relationships as each other.
- This approach is one way of illustrating how the handling of algebraic expressions can grow out of familiarity with handling numbers. Working first with the original 1 to 9 number square helps to provide insights into solving the algebraic square.

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3 The next activity involves collecting like terms in the context of addition grids. Once again, reasoning is involved.

Complete the addition squares and answer the questions on **Resource 3c, Addition squares**.

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4 Compare your explanations and justifications on Resource 3c with those below.

1 *An explanation and justification of the solution to the second of the first pair of problems might go like this.*

For the top left entry:

$97 + \square = 126$

The number in the box must be 29, since 126 – 97 = 29.

For the bottom left entry:

$29 + \square = 178$

This time, the number in the box must be 149, since 178 – 29 = 149 …
… and so on.

2 *An explanation and justification of the solution to the second of the second pair of problems might go like this.*

For the top left entry:

$(3f + 4g) + (\square + \square) = 4f - 3g$

The terms in the boxes must be f and $^{-}7g$, giving the expression f + $^{-}7g$ or f – 7g.

For the bottom left entry:

$(f - 7g) + (\square + \square) = 3f - 5g$

This time, the terms in the boxes must be 2f and 2g, giving the expression 2f + 2g …
… and so on.

Now consider the questions on **Resource 3d, Reflections on addition squares**.

5 Compare your answers to the questions on Resource 3d with those below.

- What do you notice about the diagonals in the addition squares on Resource 3c?
  Each diagonal has the same total.
- Prove that this must always be the case for a 2 by 2 addition square.
  A proof that each diagonal in a 2 by 2 addition square must have the same total can be derived from a square like this.

$+$	a	b
c	$a + c$	$b + c$
d	$a + d$	$b + d$

The total of the cells on the diagonal from top left to bottom right is
$(a + c) + (b + d) = a + b + c + d$

The total of the cells on the diagonal from top right to bottom left is
$(b + c) + (a + d) = a + b + c + d$

- What are the advantages of moving explicitly from numbers to algebra in activities like these?
  Pupils can draw the parallels between the arithmetic and algebraic processes. Their understanding of what happens with numbers helps them to understand and generalise what happens with algebraic expressions.

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6 Complete the addition squares and answer the questions on **Resource 3e, More addition squares**. This time you are given the 'output' expressions and need to think about what the missing 'input' expressions might be.

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7 Compare your answers to the questions on Resource 3e with those below.

- How many solutions are there to each of the addition square puzzles?
  All except the third example have an infinite number of solutions. The third example has no solution.
- Prove your statements about the number of solutions.
  Proofs about the number of solutions might go like this.

  The third example has no solution, since in a complete 2 by 2 addition square the diagonals must have the same total. In this example, each diagonal has a different total.

  In the first, second and fourth puzzles, there is a limitless choice for the first and, therefore, subsequent inputs. For example, put an unrelated term (say z) as an input that can take any value. This produces a workable solution.

$+$	$3a + 5b - z$	$5a + 3b - z$
z	$3a + 5b$	$5a + 3b$
$z - a + 2b$	$2a + 7b$	$4a + 5b$

Part 4 Approaches to algebra in the classroom

1 Study the yearly teaching programmes for algebra, in Framework section 2.

As you do so, identify where objectives linked to the simplification or transformation of algebraic expressions occur.

2 Now study the supplement of examples, Framework section 4:

- pages 116–121, which focus on the learning objective 'Simplify or transform algebraic expressions'.
- pages 30–35, which focus on using and applying mathematics.

As you study the examples, identify more opportunities for linking the rules of algebra to those of number in Key Stage 3 mathematics lessons. Make a note of these examples in your personal file.

3 Consider and make notes on the questions on **Resource 3f, Approaches to algebra in the classroom.**

Part 5 Summary

1 Some important principles in the teaching of algebra in Key Stage 3 are as follows.

- Key Stage 3 pupils need to develop understanding that algebra is a way of generalising.
  It helps pupils when they are aware that what works with numbers works also with algebraic expressions.
- There are stimulating activities that can be developed from work with numbers and which allow Key Stage 3 pupils to practise simplifying or transforming algebraic expressions.
  The Framework illustrates different examples that can be used in this way. Many of these activities also involve algebraic reasoning.
- Key Stage 3 pupils should be asked to explain their solutions to algebraic problems and to justify their mathematical reasoning.
  Pupils working confidently at level 5 or above should be asked to prove their results.

2 Look back over the notes you have made during this module. Have you identified the most important things that you may need to consider and adopt in your planning and teaching of algebra?

Use **Resource 3g, Summary and further action on Module 3**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and points to discuss with your head of department.

3 If you are interested in reading more about the teaching of algebra in secondary schools, download *Teaching and learning algebra pre-19*, a joint report from the Royal Society and the Joint Mathematical Council, from www.royalsoc.ac.uk/document.asp?id=1910. You may also find it useful to download and study *Interacting with mathematics in Key Stage 3: Constructing and solving linear equations* http://www.standards.dfes.gov.uk/keystage3/respub/ma_interlin.

Resource 3a An algebra loop card game

Copy, cut out and shuffle up the 18 cards below. Spread out the cards on a flat surface so that you can see them all.

Take one of the cards at random and place it to the left of you. Find the answer to the question on this card and place it to the right of the first card to form a line.

Carry on until you have used all 18 cards. The last card on the right of the line should link back to the first card on the left.

I have $3a + 2b$ What is $4b$ less?	I have $2a + 4b$ What is $4a$ more?	I have $3a - 5b$ What is $7b$ more than this?
I have $6a - 3b$ What is **one third** of this?	I have $3a + 4b$ What is $9b$ less than this?	I have $7a - 2b$ What is $a + b$ less?
I have $3a + 5b$ What is $a + b$ less than this?	I have $^{-}a - b$ What is $3b$ more?	I have $3a + 3b$ What is $2b$ more than this?
I have $2b$ What is $a + b$ more than this?	I have $2a - b$ What is **double** this?	I have $2b - a$ What is a more than this?
I have $6a + 4b$ What is a more than this?	I have $7a + 4b$ What is $4a$ less than this?	I have $a + 3b$ What is $2a$ more than this?
I have $^{-}2a - 2b$ What is **half** of this?	I have $3a - 2b$ What is $4a$ more?	I have $4a - 2b$ What is $6a$ less than this?

Resource 3b An algebraic magic square

Copy, cut out and shuffle up the nine cards below.

Arrange the cards to form a magic square.

$a + 4b$	$2a + b$	$8a + 3b$
$5a + 2b$	$7a + 6b$	$7b$
$6a + 9b$	$4a + 5b$	$3a + 8b$

Resource 3c Addition squares

1 Complete these two addition squares.

+	8	14
3	…	…
17	…	…

+	…	…
97	126	…
…	178	537

What number skills are being practised?

What other mathematical skills are involved?

How would you explain and so justify your solution to someone else?

2 Complete these two addition squares.

$+$	$2c + 3d$	$8c + 2d$
$4c + 5d$	…	…
$3c + d$	…	…

$+$	…	…
$3f + 4g$	$4f - 3g$	…
…	$3f - 5g$	$2f - g$

What algebraic skills are being practised?

What other mathematical skills are involved?

How would you explain and so justify your solution to someone else?

Resource 3d Reflections on addition squares

What do you notice about the diagonals in the addition squares on **Resource 3c**?

Prove that this must always be the case for a 2 by 2 addition square.

What are the advantages of moving explicitly from numbers to algebra in activities like these?

Resource 3e More addition squares

Complete these addition squares.

+	…	…
…	$3a + 5b$	$5a + 3b$
…	$2a + 7b$	$4a + 5b$

+	…	…
…	$2g + 5h$	$6g - 3h$
…	$4h - g$	$3g - 4h$

+	…	…
…	$8t - 3u$	$5t - 4u$
…	$15t - u$	$2t + 6u$

+	…	…
…	$5a^2 + 8ab$	$6a^2 - 3ab$
…	$2a^2 + 10ab$	$3a^2 - ab$

When you have worked on each of the puzzles for a few minutes, make some notes on your answers to the questions on the next page.

How many solutions are there to each of the addition square puzzles?

Prove your statements about the number of solutions to each puzzle.

Resource 3f Approaches to algebra in the classroom

Consider the examples of activities that you have tried out during your study of this module, and the examples that you have looked at in the *Framework for teaching mathematics: Years 7, 8 and 9*.

What objectives from the yearly teaching programmes, and for which year groups, do the activities in this module address?

What other activities from the supplement of examples could you incorporate in lessons to teach these objectives?

How could you adapt or extend the activities for other Key Stage 3 classes?

Consider the questions in this module that guided you through the activities and helped you to reflect on them. Look back through the module and identify the questions that you could incorporate into your questioning of pupils.

How would you introduce activities like these into your classroom? What modifications, if any, would you need to make to your planning, questioning styles or classroom organisation?

Resource 3g Summary and further action on Module 3

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching of algebra.

List two or three key points that you have learned.

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-
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List two or three points to follow up in further study.

-
-
-

List two or three modifications that you will make to your planning or teaching of algebra.

-
-
-

List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

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-
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MODULE

4

Making links in algebra

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- considers the development of work on sequences, functions and graphs in Key Stage 3;
- considers links across different aspects of algebra and links with other strands of the mathematics curriculum;
- discusses the use of challenging activities to develop pupils' algebraic reasoning and use of algebra in solving problems.

CONTENT

The module is in five parts.

1 Using algebra to solve problems

2 Generalising

3 Linking sequences, functions and graphs

4 Looking at a lesson

5 Summary

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- The *Framework for teaching mathematics: Years 7, 8 and 9*
- Video sequence 2, a Year 8 algebra lesson, from the CD-ROM accompanying this module
- The resource sheets at the end of this module:

  4a Sequences, functions and graphs: definitions and examples

  4b Fibonacci chains

  4c Generalising

  4d Squares in a cross

  4e Graphs of linear functions

  4f Julie's lesson

  4g Summary and further action on Module 4

Desirable

- *Interacting with mathematics in Key Stage 3: Constructing and solving linear equations*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_interlin
- *Teaching and learning algebra pre-19*, a joint report from the Royal Society and the Joint Mathematical Council
  http://www.royalsoc.ac.uk/document.asp?id=1910

STUDY TIME

Allow approximately 90 minutes.

Part 1 Using algebra to solve problems

1 Algebra in Years 7 to 9 includes equations, formulae and identities, and sequences, functions and graphs. Module 3 focuses on some aspects of equations, formulae and identities. Module 4 focuses on sequences, functions and graphs. During this module, you will watch a Year 8 lesson linking algebra and geometry.

2 You are probably familiar with the definitions of a sequence, a function and a graph. By the end of Key Stage 3, pupils need to be aware of and understand these definitions. Check your understanding against the definitions on **Resource 4a, Sequences, functions and graphs: definitions and examples**.

3 Pupils have been developing their ideas about pattern in number throughout Key Stages 1 and 2. One aspect of this work relates to number properties and sequences, such as:

- one more than a multiple of 3;
- figurate numbers such as square numbers and triangular numbers;
- Fibonacci numbers.

In the problems on **Resource 4b, Fibonacci chains**, the number sequences have similar properties to the Fibonacci sequence – that is, each term is the sum of the previous two terms. Solve these problems and make some notes on a method that can be used to solve them.

4 While the answer to the first problem on Resource 4b is easy to spot, you probably used algebraic methods to solve the other problems.

All the sequences can be generalised in this form:

$a,\quad b,\quad a+b,\quad a+2b,\quad 2a+3b,\quad 3a+5b,\quad 5a+8b,\quad \ldots,$

where a is the first term and b is the second term. This allows the following equations to be formulated and solved:

$6 + 3b = 18$	3	4	7	11	18		
$5.66 + 3b = 25.91$	2.83	6.75	9.58	16.33	25.91		
$12 + 5b = 36$	4	4.8	8.8	13.6	22.4	36	
$30 + 8b = 4$	6	$^{-}3.25$	2.75	$^{-}0.5$	2.25	1.75	4

The four examples show that changing the first and last terms in sequences of this type alters the level in difficulty, so that an activity for pupils that is based on this task can readily be differentiated. For example, the first problem would be suitable for pupils working confidently at level 4, whereas the other three problems are more suitable for pupils working at level 5 or level 6.

5 Now study the supplement of examples, Framework section 4:

- pages 144–159, which focus on sequences;
- pages 6–9, which focus on problems involving number and algebra.

As you study the examples, identify more opportunities for using algebra to solve problems in Key Stage 3 mathematics lessons. Make a note of these examples in your personal file.

Part 2 Generalising

1 In the Fibonacci sequence problems in Part 1, it was possible to express each term of the sequences in a general form by using letters to stand for numbers.

One way of introducing pupils to algebraic generalisation is to ask them to extend number patterns. Answer the questions on **Resource 4c, Generalising**, which are typical of the problems that can be given to pupils working at level 5.

2 In the first problem on Resource 4c, you have probably described the nth line in the pattern by describing it as:

$(n - 1)(n + 1) = n^2 - 1$

or as:

$n(n + 2) = (n + 1)^2 - 1$

The pattern can be extended backwards to explore multiplication of negative numbers, since any integer, positive or negative, can be substituted for n.

$$\begin{aligned}
1 \times 3 &= 2^2 - 1\\
0 \times 2 &= 1^2 - 1\\
(^-1) \times 1 &= 0^2 - 1\\
(^-2) \times 0 &= (^-1)^2 - 1\\
(^-3) \times (^-1) &= (^-2)^2 - 1
\end{aligned}$$

What happens if fraction or decimal values are substituted for n? Is it still the case that $(n - 1)(n + 1) = n^2 - 1$?

An equation like $n^2 - 1 = (n - 1)(n + 1)$ that holds true for all possible values of the variables is called an identity.

The second problem on Resource 4c has a connection with the first problem, in that each of the four numbers 899, 3599, 10 403, 359 999 is 1 less than a perfect square. It can therefore be expressed in the form $n^2 - 1$, which factorises as $(n - 1)(n + 1)$. This helps to find the solutions:

$899 = 30^2 - 1 = 29 \times 31$
$3599 = 60^2 - 1 = 59 \times 61$
$10\,403 = 102^2 - 1 = 101 \times 103$
$359\,999 = 600^2 - 1 = 599 \times 601$

What happens with other values? Check that:

$(5.816)^2 - 1 = 4.816 \times 6.816$

and that:

$(^-15.216) \times (^-13.216) + 1 = (^-14.216)^2$

Giving pupils opportunities to explore generalities like these helps them to develop an understanding of the power of algebra.

Part 3 Linking sequences, functions and graphs

1 Connections between sequences, functions and graphs are often not given enough emphasis in Key Stage 3 mathematics lessons.

Pupils are often introduced to functions through ‘number machines’ or ‘function machines’. In this first example, the rule is given, along with some input numbers. Pupils soon learn to work backwards intuitively from the output numbers.

in $\longrightarrow$ $\boxed{\text{multiply by 4 then add 1}}$ $\longrightarrow$ out

in	1	7	200	2.5			–2
out					121	2	

In Year 7 pupils are expected to begin to use algebraic notation:

$n \rightarrow 4n + 1$ or $y = 4x + 1$

The mapping can be separated out into single operation machines:

$x \longrightarrow \boxed{\times 4} \longrightarrow \boxed{+\,1} \longrightarrow y$

By Year 8, pupils should be able to transform this using inverse operations:

$x \longleftarrow \boxed{\div 4} \longleftarrow \boxed{-\,1} \longleftarrow y$

or: $x = \dfrac{y-1}{4}$

This links closely to the second form of machine, which is of the ‘What went in?’ type.

in $\longrightarrow$ $\boxed{\text{multiply by 2 then add 3}}$ $\longrightarrow$ out

in							
out	7	25	3	55	863	3.1	2

The function for this machine is $y = 2x + 3$ and its inverse is $x = \frac{1}{2}(y - 3)$.

Both types of machine, once introduced, would provide useful number practice in an oral and mental session. The examples show how operations on different numbers can be targeted for practice, and so questions can be matched to the stage of development of individual pupils.

A third type of number machine has input and corresponding output numbers and the function has to be found.

in $\longrightarrow$ $\boxed{}$ $\longrightarrow$ out

in	3	4	6	10	1.7		–5
out	9	13	21	37		0	

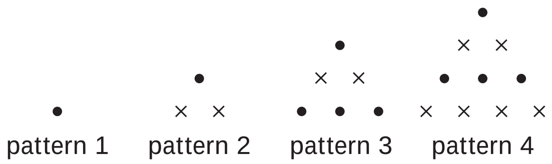
In this machine, the function is $y = 4x - 3$ and $x = \frac{1}{4}(y + 3)$.

What number is unchanged by this machine? Or, to put this question another way, for what number are the input and output numbers the same? What are the ‘stay the same’ numbers for the first two types of number machines?

A possible investigation for pupils is to explore ‘stay the same’ numbers for different number machines.

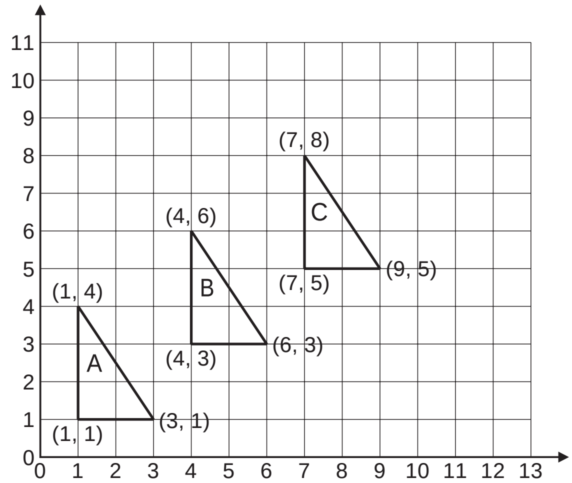
2 Another context in which pupils will meet mappings such as these is in work on number sequences. Work on number sequences will have started in Key Stage 2. The two questions below are level 4 questions from the Key Stage 2 National Curriculum tests.

This series of patterns grows in a regular way.



How many **dots** would be in pattern 5?

How many **crosses** would be in pattern 5?



Write the coordinates of the next triangle in the sequence.

Look at the number sequence on **Resource 4d, Squares in a cross**, and answer the accompanying questions.

3 The function related to *Squares in a cross* is represented by the equation $y = 4x + 1$.

The large numbers in the table are beyond pupils’ capacity to draw and count. This encourages them to move from particular cases to the general. Key Stage 3 pupils should first be asked to find the rule in words, to provide a step towards the algebra, and then to suggest a way of checking on its accuracy.

Two levels of generalisation emerge from these types of spatial patterns. For pupils, it is easier to spot links between successive terms, for example ‘It goes up in fours’, than to relate a term to its position in the sequence.

It helps pupils if you encourage them to justify and explain why a rule or relationship works in the context of the situation, relating back to the diagrams and not just to the pattern of numbers. One way of doing this is to ask them to calculate particular terms in the sequence. For example, in *Squares in a cross*, the 10th cross needs $10 \times 4 + 1$, or 41 squares, and the 100th cross needs $100 \times 4 + 1 = 401$ squares. From the particular examples, they are able to see that, in general, $y = 4x + 1$.

4 With number sequences based on spatial patterns like these, the values of the variables are whole-number values only. In a true algebraic relationship, the variables can take any values on a continuous scale. Graphs of number patterns should really be a set of separate points, but in order to look at the algebraic relationship we usually join the points as though they represent a continuous function.

Look at **Resource 4e, Graphs of linear functions**. One of the graphs represents the function $y = 4x + 1$. Identify which graph it is and then find the equations of the other graphs.

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5 Pupils in Year 7 are expected to draw graphs such as those on Resource 4e. In Year 8, pupils are introduced to ideas of gradient and intercept.

Compare your answers to the questions on Resource 4e with those below.

A $x + y = 4$
B $y = 4x + 1$
C $y = 4x - 3$
D $y = \frac{1}{2}x + 3$
E $y = 2x + 3$

The graph for which the point of intersection with each line would be the 'stay the same' number for that function is the line $y = x$.

Linking elements of algebra together is part of the algebraic reasoning that needs to be developed throughout Key Stage 3. Pupils need to gain insight into the power and purpose of algebra as well as learning algebraic techniques.

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6 Study the supplement of examples, Framework section 4, pages 160–177, which focus on functions and graphs.

As you study the examples, identify more opportunities for linking work on sequences, functions and graphs in Key Stage 3 mathematics lessons. Note these examples in your personal file.

Part 4 Looking at a lesson

1 Algebraic ideas and reasoning can support pupils' understanding of other areas of mathematics. There are, for example, strong links between algebra and geometric reasoning.

Get ready to watch **Video sequence 2, a Year 8 algebra lesson**, which exemplifies these links. The teacher is Julie.

You will notice that although Julie makes it clear early on that the letters represent the areas of the shapes (and not 'the shapes'), the pupils are not always so precise in their language. It is important that pupils understand that the letters represent values which, although unknown, can be handled like known numbers.

As you watch the lesson, focus particularly on the questioning styles adopted by Julie in her direct teaching. Use **Resource 4f, Julie's lesson**, to make notes.

The video sequence lasts about 11 minutes.

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2 When you have finished watching, spend a few minutes completing the notes you have made on **Resource 4f**. Then think about how Julie's approach compares with what you would have done.

Part 5 Summary

1 Understanding the links between sequences, functions and graphs is a cornerstone of Key Stage 3 mathematics. Teaching has to help pupils to appreciate that algebra allows them to represent and explore general relationships and that this is more powerful than looking only at specific cases.

Pupils need opportunities to use their algebraic skills in problem solving in order to:

- increase their awareness of when and how algebra can be useful;
- improve their knowledge of algebraic conventions;
- deepen their understanding of algebraic rules;
- practise their use of algebraic techniques.

Most importantly, they need opportunities to see how algebra can provide insights into the underlying situation that the algebra is modelling.

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2 Look back over the notes you have made during this module. Have you identified what you may need to consider and adopt in your planning and teaching of algebra?

Use **Resource 4g, Summary and further action on Module 4**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and points to discuss with your head of department.

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3 As mentioned in Module 3, if you are interested in reading more about the teaching of algebra in secondary schools, download *Teaching and learning algebra pre-19*, a joint report from the Royal Society and the Joint Mathematical Council, from http://www.royalsoc.ac.uk/document.asp?id=1910.

If you have not already done so, you could also download and look at the Key Stage 3 Strategy's *Interacting with mathematics in Key Stage 3: Constructing and solving linear equations,* from http://www.standards.dfes.gov.uk/keystage3/respub/ma_interlin.

Resource 4a Sequences, functions and graphs: definitions and examples

Sequences

A **sequence** is an ordered succession of terms formed according to a rule. There can be a finite or infinite number of terms.

The sequences most commonly considered in Key Stage 3 mathematics have:

- an identifiable mathematical relationship between the value of a term and its position in the sequence; and/or
- an identifiable mathematical rule for generating the next term in the sequence from one or more existing terms.

Examples
The squares of the integers: 1, 4, 9, 16, 25, …
The Fibonacci sequence: 1, 1, 2, 3, 5, 8, …

Functions

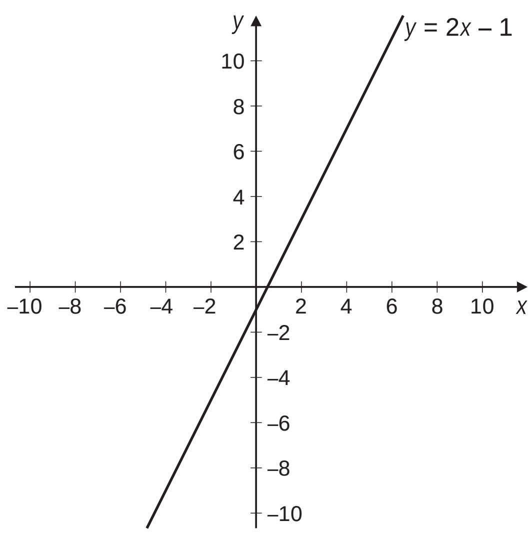
A **function** is a rule that associates each term of one set of numbers with a single term in a second set. The relationship can be written in different ways.

Example
$x \rightarrow 2x - 1$
$y = 2x - 1$

Graphs

A **graph** of a function is a diagram that represents the relationship between two variables or sets of numbers.

Example



Resource 4b Fibonacci chains

All these number chains have similar properties to the Fibonacci sequence – that is, each term is the sum of the previous two.

Find the missing terms.

3	…	…	…	18

2.83	…	…	…	25.91

4	…	…	…	…	36

6	…	…	…	…	…	4

Use this space for any working that you want to do.

Explain how to solve problems like these.

Resource 4c Generalising

1 Consider this pattern:

$1 \times 3 = 2^2 - 1$

$2 \times 4 = 3^2 - 1$

$3 \times 5 = 4^2 - 1$

$4 \times 6 = 5^2 - 1$

a. What will the next two lines be?

b. What will the 10th line be?

c. What will the 100th line be?

d. If I wanted to know what a particular row will be, say the *n*th row, how could you tell me?

2 Find a pair of factors of:

a. 899

b. 3599

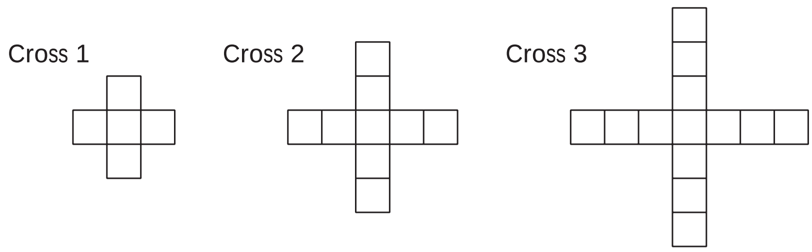
c. 10 403

d. 359 999

Make up two similar questions.

Resource 4d Squares in a cross

Fill in the missing values in the table by studying the patterns.



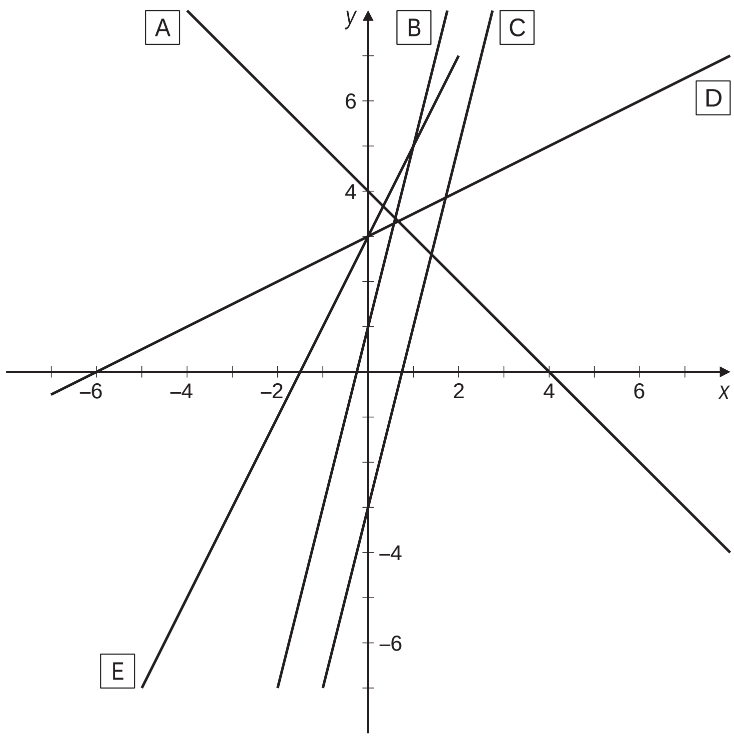
Number of cross	x	1	2	3	4	5	10	50		200
Number of squares	y								1001	

What is the rule which links the number of squares in any cross (y) to its position in the sequence (x)?

Explain why this is the rule by referring to the properties of the shapes.

Resource 4e Graphs of linear functions

Which line represents the function $y = 4x + 1$?



Find the equations of the other lines.

Line	Equation
A	
B	
C	
D	
E	

What other graph would you need to draw so that its point of intersection with each of the other lines would be the ‘stay the same’ number for that line’s function?

Resource 4f Julie's lesson

How do Julie's questions help pupils to visualise and explain their solutions in different ways?

How does the activity help Julie's pupils to make links between algebra and geometry?

How valuable was the final plenary of the lesson in establishing those links?

Resource 4g Summary and further action on Module 4

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching of algebra.

List two or three key points that you have learned.

-
-
-

List two or three points to follow up in further study.

-
-
-

List two or three modifications that you will make to your planning or teaching of algebra.

-
-
-

List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

-
-
-

MODULE

5

Geometrical reasoning 1

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- looks at approaches to developing pupils' visualisation and geometrical reasoning skills;
- considers progression towards geometric proof.

CONTENT

The module is in five parts.

1 Introduction

2 Conventions, definitions and derived properties

3 Deriving properties

4 Looking at a lesson on geometrical reasoning

5 Summary

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- The *Framework for teaching mathematics: Years 7, 8 and 9*
- Video sequence 3, a Year 8 geometry lesson, from the CD-ROM accompanying this module
- The resource sheets at the end of this module:

  5a Visualisation activities

  5b Conventions and definitions

  5c Deriving properties

  5d More derivations

  5e Bola's lesson

  5f Examples from National Curriculum tests for Key Stages 2 and 3

  5g Summary and further action on Module 5

Desirable

- *Year 9 geometrical reasoning: mini-pack* http://www.standards.dfes.gov.uk/keystage3/respub/ma_intery9geom
- *Teaching and learning geometry 11–19*, a joint report from the Royal Society and the Joint Mathematical Council, available at: http://www.royalsoc.ac.uk/downloaddoc.asp?id=1191
- The QCA Mathematics glossary for teachers in Key Stages 1 to 4, available at http://www.qca.org.uk/279_2104.html

STUDY TIME

Allow approximately 90 minutes.

Part 1 Introduction

1 As pupils move through Key Stages 1 and 2, they progress from a mainly practical and experiential approach to shape and space to a more structured, formalised approach.

The main aim in Key Stage 3 is for pupils to develop their knowledge of geometrical ideas and use it to support geometrical reasoning. This approach lays the foundations for a more formal approach to geometrical proof in Key Stage 4. (A mathematical proof involves establishing the truth of a statement by rigorous logical argument.)

Geometrical reasoning and proof have had greater emphasis in the National Curriculum since 2000 than previously. This emphasis is reflected in the *Framework for teaching mathematics: Years 7, 8 and 9*.

Geometrical reasoning is the focus for this and the next module.

---

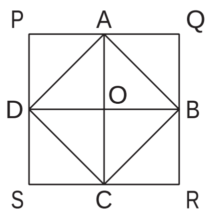
2 Try the two activities on **Resource 5a, Visualisation activities**.

---

3 The two activities on Resource 5a aim to develop visualisation, geometrical reasoning and justification. These and similar activities make useful oral and mental starters for geometry lessons. Since there is often more than one way of justifying the result, you can ask pupils to compare their justifications by describing them to a partner or to the whole class.

Compare your arguments with those below.

Midpoints



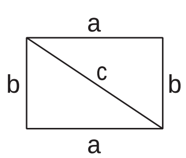
ABCD is a square with an area that is half the area of square PQRS.

A justification:

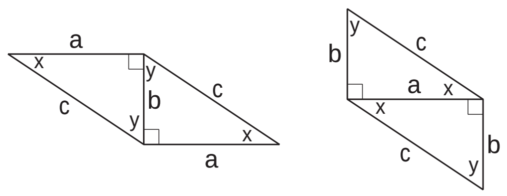
AC and BD are the perpendicular bisectors of the sides of the square PQRS. They are equal in length and bisect each other at right angles. Since AC and BD are the diagonals of ABCD, it follows that ABCD is a square.

The eight triangles formed by the construction lines are congruent and so equal in area (two sides and the included right angle are equal). It follows that the area of square ABCD, which is formed from four of the triangles, is half the area of square PQRS.

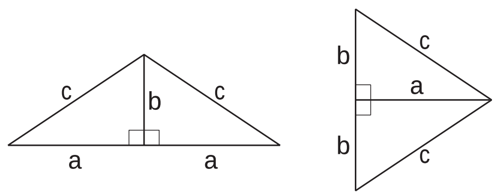
Changing shapes



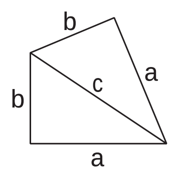
The diagonal of the rectangle creates two identical right-angled triangles. These can be used to make five possible shapes in addition to the original rectangle: two different parallelograms, two different isosceles triangles and a kite.



In these two shapes, it can be shown that opposite sides are equal and opposite angles are equal, making them parallelograms.



In these two shapes, it can be shown that two sides are equal, and that the third side is a straight line, making them isosceles triangles. (It is necessary to prove that the third side is a straight line, otherwise the shape would be a quadrilateral with two adjacent sides equal.)



In this shape, it can be shown that two pairs of adjacent sides are equal, making it a kite.

---

4 Now look in more detail at aspects of geometrical reasoning in the *Framework for teaching mathematics: Years 7, 8 and 9*. Study the teaching programmes for Years 7, 8 and 9, Framework section 3, pages 7, 9 and 11.

As you study the teaching programmes, jot down in your personal file some of the words that illustrate the emphasis on geometrical reasoning.

As well as several references to 'explaining reasoning', there are objectives in the Year 9 teaching programme, and in the Year 9 extension programme, which rely on fairly sophisticated thought, such as:

- distinguish between conventions, definitions and derived properties;
- distinguish between practical demonstration and proof (Year 9 extension).

This module focuses in particular on the progression through Key Stage 3 that builds up to these objectives in Year 9.

Part 2 Conventions, definitions and derived properties

1 Study the examples of conventions and definitions on **Resource 5b, Conventions and definitions**. Are you familiar with these?

Other properties of angles and shapes can be derived from these definitions. For example, it is possible to prove, rather than define, that 'vertically opposite angles are equal' or that 'alternate angles are equal'.

---

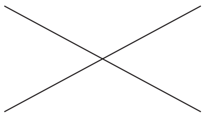
2 Sue Waring, in her book *Can you prove it?*, published by The Mathematical Association, identifies four possible stages for pupils as they work towards a formal proof:

Stage 1 Convince yourself (mental justification)
Stage 2 Convince a friend (oral justification)
Stage 3 Convince a pen-friend (informal written justification)
Stage 4 Convince your mathematics teacher (more formal written justification)

These four stages are illustrated below in relation to a proof that 'vertically opposite angles are equal'.

Stage 1

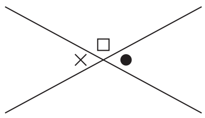
Stage 1 involves convincing yourself.



For example, you might think: 'Those two angles are on a straight line and so are those two angles. So I can take that big angle away from both those straight angles and the two remaining little angles must be equal.'

Stage 2

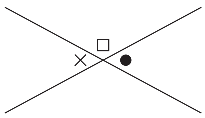
Stage 2 involves convincing a friend.



For example, you might say: 'The angle marked with a circle and the angle marked with a square add up to 180°. The same is true for the angle marked with a cross and the one marked with the square. So this angle (point to the circle) must equal this one (point to the cross).'

Stage 3

Stage 3 involves an informal written justification, which might go like this.



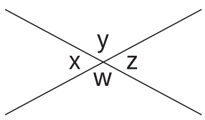
● + □ = 180

× + □ = 180

So × = ●

Stage 4

Stage 4 involves a formal written justification, which might go like this.



$x + y = 180$ (angles on a straight line)

$y + z = 180$ (angles on a straight line)

$\therefore\ x + y = y + z$, giving $x = z$.

3 Now study the examples in the supplement of examples, Framework section 4, pages 178–179.

As you look at the examples, think about the differences between conventions, definitions and derived properties. Distinguishing between a demonstration and a proof is in the Key Stage 4 programme of study and is in the Year 9 extension teaching programme exemplified, in italics, in the last paragraph of page 179.

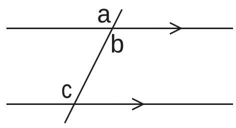
Now study the examples on pages 180–201. These illustrate the progression from informal explanation and justification to formal proof. As you work through the examples, make a note in your personal file of any that would be useful to explore with the classes that you teach.

Part 3 Deriving properties

1 Do the problems on **Resource 5c, Deriving properties**. You are given some definitions of geometric properties and, from these, must deduce some further geometric properties. Aim to produce a stage 4 formal proof wherever possible.

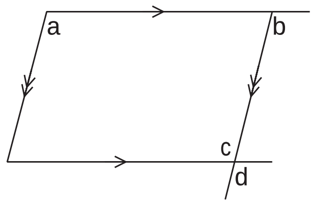
2 Here are some possible arguments that can be used for the problems on Resource 5c.

1



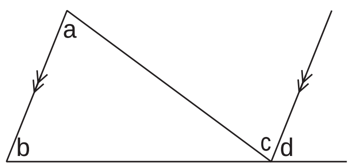
Take any pair of alternate angles, for example a and b.
$a = b$ (vertically opposite angles)
$a = c$ (corresponding angles)
$\therefore\ b = c$
So alternate angles between parallel lines are equal.

2



Take any pair of opposite angles, for example a and c.
$a = b$ (corresponding angles)
$b = c$ (alternate angles)
$\therefore\ a = c$
So the opposite angles of a parallelogram are equal.

3



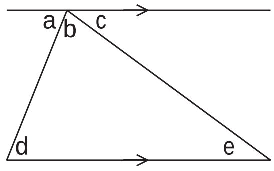
Extend one side of the triangle and construct a line through one vertex parallel to the opposite side, as shown.
$a = c$ (alternate angles)

$b = d$ (corresponding angles)

$\therefore\ a + b = c + d$

So the exterior angle of a triangle equals the sum of the interior opposite angles.

4



$d = a$ (alternate angles)

$e = c$ (alternate angles)

$\therefore\ d + b + e = a + b + c = 180$ (angles on a straight line)

So the sum of the interior angles of a triangle is 180°.

---

3 Now try the problems on **Resource 5d, More derivations**.

To find the sum S of the interior angles of an n-sided polygon, identify a point O inside the polygon. Join O to each of the vertices of the polygon, forming n triangles. S can be regarded as the sum T of the angles of all n triangles, less the sum A of the angles around point O. Since A is 360° or 4 right angles, S can be calculated in two ways:

- since each of the n triangles has an angle sum of 180°, T is $180n$°, and $S = 180n - 360$ degrees, which is $(n - 2) \times 180$°;
- since each of the n triangles has an angle sum of 2 right angles, T is $2n$ right angles, and $S = 2n - 4$ right angles.

Part 4 Looking at a lesson on geometrical reasoning

1 In this part of the module, you will have a chance to consider the use of ICT in developing geometrical reasoning.

First study the examples on pages 180–181 of the supplement of examples, Framework section 4. These examples refer to the use of acetate sheets for an overhead projector or use of computer software. As you study the examples, think about the relative advantages of one medium over the other. Note in your personal file any examples that would be useful to explore with the classes that you teach.

---

2 Get ready to watch **Video sequence 3, a Year 8 geometry lesson**. The teacher is Bola.

Bola is using dynamic geometry software to begin to develop her pupils' ideas of proof. In the first part of the video sequence, Bola discusses with the class how to name angles and demonstrates the equality of vertically opposite angles. In the second part, which is much later in the lesson, she questions pupils about their proof that the sum of the angles of a triangle is 180°.

As you watch the video, consider the usefulness of dynamic geometry software, focusing on the questions on **Resource 5e, Bola's lesson**.

The video sequence lasts about 5 minutes.

When you have finished watching, spend a few minutes completing the notes you have made on Resource 5e. Then think about how Bola's approach compares with what you would have done.

---

3 Study the geometric problems in the examples on using and applying mathematics in the supplement of examples, Framework section 4, pages 14–17. These include examples that make use of computer software or acetate sheets. They also include more examples in which properties of shapes have to be derived by geometrical reasoning.

Add to the notes in your personal file one or more problems that would be useful to offer to the classes that you teach.

Part 5 Summary

1 It is important that Key Stage 3 pupils appreciate the differences between geometric conventions, definitions and derived properties.

As they use and apply their developing knowledge of geometrical properties, pupils in Key Stage 3 should move from informal justifications of their arguments to more formal written proofs.

---

2 Look at **Resource 5f, Examples from National Curriculum tests**. What definitions would pupils need to know in order to answer the questions?

For each question, think about the kinds of informal or formal arguments that you would expect pupils to give to justify their reasoning.

---

3 Look back over the notes you have made during this module. Have you identified what you may need to consider and adopt in your planning and teaching of geometry?

Use **Resource 5g, Summary and further action on Module 5**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and points to discuss with your head of department.

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4 You may find it interesting to read the article by Paul Chambers on 'Teaching Pythagoras' theorem' from the September 1999 issue of *Mathematics in School*, published by The Mathematical Association, 259 London Road, Leicester LE2 3BE.

If you are interested in reading more about the teaching of geometry in secondary schools, read *Teaching and learning geometry 11–19*, a joint report from the Royal Society and the Joint Mathematical Council. This report reiterates the centrality of geometry to the mathematics curriculum and how important it is that this branch of the subject should not be neglected. Appendix 9: Proof – 'why and what?' is of particular interest. You can download the report from http://www.royalsoc.ac.uk/downloaddoc.asp?id=1191.

You could also download and look at the *Year 9 geometrical reasoning: mini-pack* from http://www.standards.dfes.gov.uk/keystage3/respub/ma_intery9geom.

Resource 5a Visualisation activities

The first part of each of these activities should be carried out without any drawing.

1 MIDPOINTS

Imagine a square.

Join the midpoints of each pair of adjacent sides.

What is the inscribed shape?

How does the area of the inscribed shape relate to the area of the original square?

Now justify your reasoning. Draw a sketch and use informal language if you wish.

[continued on the next page]

2 CHANGING SHAPES

Imagine a (non-square) rectangle.

Cut it along one of its diagonals so that you have two shapes. Call these shapes A and B.

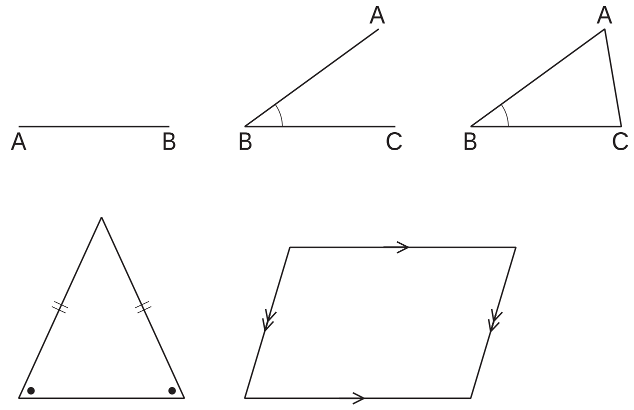
Visualise the different shapes you can make from A and B by matching sides of the same length.

Now sketch each of your new shapes and write its name. For each shape, state the geometrical facts you are using to justify your answer.

Resource 5b Conventions and definitions

CONVENTIONS

Labelling

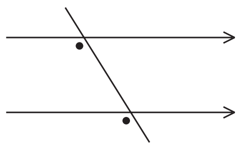


Notation

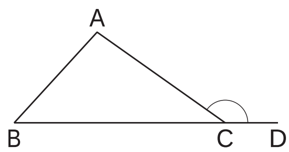
△ triangle ∠ angle ∴ therefore // is parallel to

DEFINITIONS

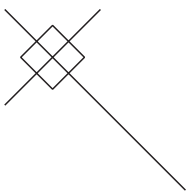
Corresponding angles lie on the same side of a transversal and on corresponding sides of a pair of lines. If the two lines are parallel, the corresponding angles are equal.



An **exterior angle** of a polygon is the angle between a side and an extension of an adjacent side. In this example, ∠ACD is an exterior angle of △ABC.



Perpendicular lines intersect at right angles.



Resource 5c Deriving properties

You are given the following facts:

- the angles on a straight line are supplementary, i.e. they add up to 180°;
- corresponding angles are equal;
- vertically opposite angles are equal;
- opposite sides of a parallelogram are parallel.

Use some or all of these facts, and constructions where necessary, to prove the following in the order 1, 2, 3, 4.

1 Alternate angles are equal.

2 Opposite angles of a parallelogram are equal.

3 The exterior angle of a triangle is equal to the sum of the interior opposites.

4 The angles of a triangle add up to 180°.

Resource 5d More derivations

Look at the Year 9 examples on page 183 of the supplement of examples, Framework section 4.

Derive the formula for the sum of the internal angles of an n-sided polygon as $(n - 2) \times 180°$.

Think of an alternative argument that would lead to '$(2n - 4)$ right angles'.

Resource 5e Bola's lesson

While watching the short video extracts of Bola working with her Year 8 class, consider and make notes on the questions below.

How does the dynamic geometry software facilitate demonstration of a given fact?

How does the dynamic geometry software facilitate proof of a geometrical property?

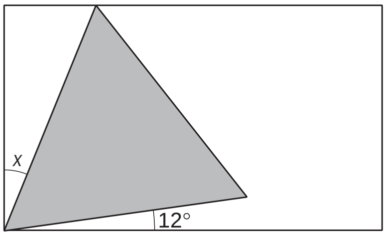
What other aspects of geometrical reasoning could be enhanced by giving pupils the opportunity to move lines and shapes in this way?

Resource 5f Examples from National Curriculum tests

These examples are taken from the National Curriculum tests for Key Stages 2 and 3.

LEVEL 5

1 Here is an equilateral triangle inside a rectangle.

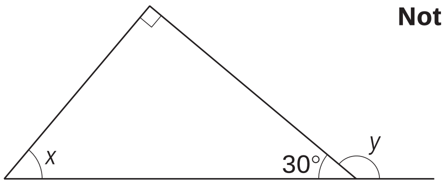


Not to scale

Calculate the value of angle *x*.
Show your working.

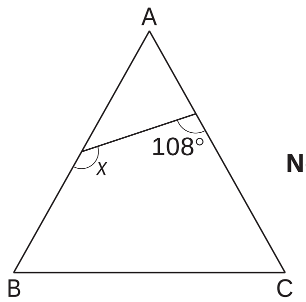
2 Look at this diagram.

Not to scale



Calculate the size of angle *x* and angle *y*.
Show your working.

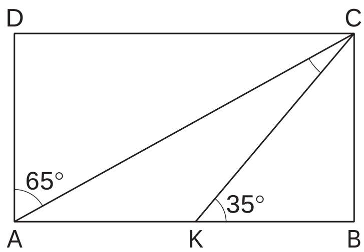
3 Triangle ABC is equilateral.



Not to scale

Calculate the size of angle *x*.
Show your working.

4 The diagram shows a rectangle.

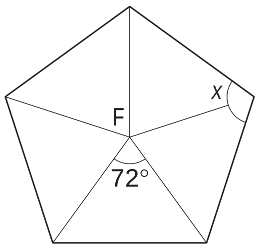


Not drawn accurately

Work out the size of angle *a*.
Show your working.

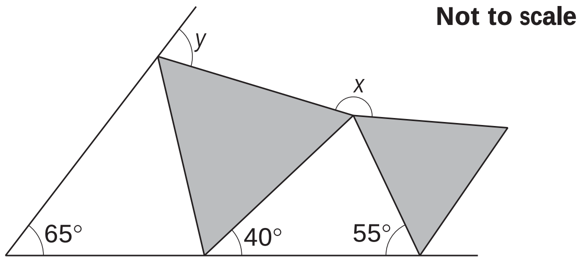
LEVEL 6

1 F is the centre of a regular pentagon.



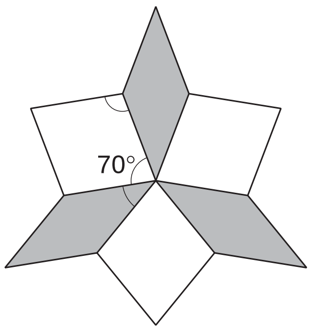
Work out the value of angle *x*.
Explain how you worked out your answer.

2 The diagram shows two shaded equilateral triangles.



Calculate the size of angle *x* and the size of angle *y*.

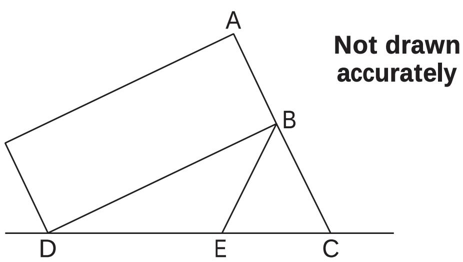
3 The shape below has three identical white tiles and three identical grey tiles.
The sides of each tile are all the same length.
Opposite sides of each tile are parallel.
One of the angles is 70°.



Calculate the size of angle *k* and angle *m*.
Show your working.

LEVEL 7

1 A rectangle just touches an equilateral triangle so that ABC is a straight line.



Show that triangle BDE is isosceles.

Resource 5g Summary and further action on Module 5

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching of geometry.

List two or three key points that you have learned.

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List two or three points to follow up in further study.

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List two or three modifications that you will make to your planning or teaching of geometry.

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List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

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MODULE

6

Geometrical reasoning 2

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- considers the connections between loci and constructions;
- discusses activities and resources to develop pupils' visual and geometrical reasoning skills.

CONTENT

The module is in five parts.

1 Introduction

2 Loci defined through distance

3 Generating loci

4 Formal constructions

5 Summary of Modules 5 and 6

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- The *Framework for teaching mathematics: Years 7, 8 and 9*
- A pencil, ruler, set square and compasses for drawing constructions
- A blank sheet of A4 paper, about 30 cm of string and a 5p coin
- The resource sheets at the end of this module:

  6a 3-D visualisation activities

  6b Moving a coin

  6c Loci problems

  6d Sample teaching unit

  6e Key Stage 3 National Curriculum tests: questions on loci

  6f Summary and further action on Module 6

Desirable

- *Year 9 geometrical reasoning: mini-pack* http://www.standards.dfes.gov.uk/keystage3/respub/ma_intery9geom
- *Teaching and learning geometry 11–19*, a joint report from the Royal Society and the Joint Mathematical Council, available at: http://www.royalsoc.ac.uk/downloaddoc.asp?id=1191
- The QCA *Mathematics glossary for teachers in Key Stages 1 to 4*, available at http://www.qca.org.uk/279_2104.html

STUDY TIME

Allow approximately 90 minutes.

Part 1 Introduction

1 Module 6 looks at ways of establishing a deeper understanding of the links between loci and constructions through visualisation and practical activity. The aim is to build on the deductive reasoning skills discussed in Module 5.

2 The two modules on geometrical reasoning concentrate mainly on examples in two dimensions. Pupils' reasoning skills also need to be developed in three-dimensional contexts.

We will therefore start with a 3-D visualisation activity. Try the two activities on **Resource 6a, 3-D visualisation activities**.

3 In the two activities on Resource 6a, the new shape formed by joining the centre of each face to the centre of adjacent faces is called the *dual* of the first shape. Compare your answers with those below.

A square-based pyramid has 5 faces, 5 vertices and 8 edges. The 5 faces will generate 5 vertices for the new shape. If we assume that the original pyramid as standing on its square base, then the dual is a smaller, different square-based pyramid, hanging upside down inside the original shape.

A cube has 6 faces, 8 vertices and 12 edges. The dual of the cube has 6 vertices and 8 faces. It is an octahedron but could also be described as two square-based pyramids sharing a common square base and pointing in opposite directions.

You probably noticed that the number of vertices and edges in the dual corresponds to the number of faces and edges in the original shape. In your follow-on questions, did you include things like: 'What is the dual of a tetrahedron?' and 'When is the dual a similar shape to the original?'

4 Now study the 3-D examples in the supplement of examples, Framework section 4, pages 198–201. Make a note in your personal file of any examples that would be useful to explore with the classes that you teach. Add to your notes two or three follow-on questions to ask pupils when they have completed the activity.

Part 2 Loci defined through distance

1 We will now look at the idea of a locus. The QCA glossary defines a **locus** as:

- the set of points that satisfy given conditions.

For example, in three dimensions, the locus of all points that are a given distance from a fixed point is a sphere.

2 Try the three problems on **Resource 6b, Moving a coin**. For these you will need a blank sheet of A4 paper, about 30 cm of string, a 5p coin, a pencil, a ruler, a set square and compasses.

3 Think about the problems you have just done. Compare your solutions with those below.

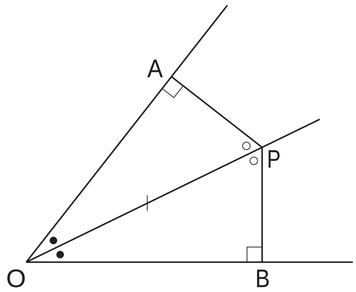
Locus 1: The locus is a line parallel to the long edge of the paper at a perpendicular distance of 5 cm from it.

Locus 2: The locus is the bisector of the right angle at the corner of the paper.

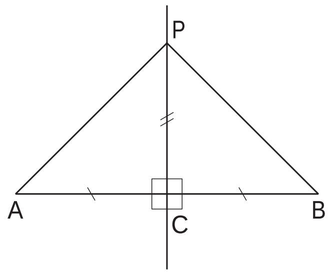
Locus 3: The locus is the perpendicular bisector of the line segment joining the two fixed points. One way of thinking about the moving point is to see it as the apex of an isosceles triangle.

How would you justify your solutions to these problems? In **locus 1**, the justification lies in the definition of parallel lines – they are always the same distance apart.

In **locus 2**, we have to justify that a point P on the bisector of angle AOB is always the same distance from each of the two arms of the angle. Informally, the angle bisector OP is a line of symmetry, so the two triangles AOP and BOP are identical. More formally, triangles AOP and BOP are congruent (two angles and the included side are the same). It follows that AP = PB.



In **locus 3**, we have to justify that a point P on the perpendicular bisector of the line joining two points A and B is always the same distance from each of the two points. Here again, we can use informal arguments based on symmetry, or we can prove that triangle PAC and PBC are congruent (two sides and the included angle). It follows that PA = PB.



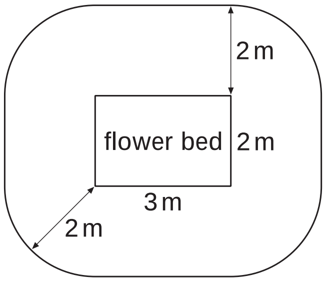
Part 3 Generating loci

1 It is often easier to think about loci by working on them practically. Exploration of problems in the classroom can sometimes be supported by asking a pupil to make moves according to the instructions given by another pupil. Other pupils can check using a ball of string whether the conditions of the original problem are being met. For example, locus 3 above could be generated by asking a pupil to move so that she remains an equal distance from each of two pupils who are standing still. Even when exact calculations are not made, it is possible to deduce properties that link with familiar shapes.

2 Try the practical investigations on **Resource 6c, Loci problems**.

When you have finished, compare your solutions to the problems on Resource 6c with the notes below.

Locus 4: It is easy to assume that the locus of points 2 metres from the flower bed is a rectangle. Closer examination of what happens at the corners is needed!



Locus 5: This generates an ellipse.

The axis of symmetry parallel to AB or 'major axis' is 10 cm long, the same as the sum of the distances of point P from A and B; the axis of symmetry perpendicular to AB is 8 cm long, found using Pythagoras' theorem.

As the pins move towards each other, the shape becomes closer to a circle. As the pins move further apart (the maximum possible distance between them is 10 cm), the shape becomes closer to the straight line AB.

Part 4 Formal constructions

1 Informal work on loci is a useful precursor to work on straight edge and compass constructions. It can provide motivation for the more formal work and encourage pupils to look for reasons why a particular method works or sometimes to discover a method for themselves.

The formal constructions are unlike the constructions that pupils have done using a ruler and protractor in Key Stage 2 and Year 7. The constructions done earlier all involve measurements in one way or another and so have a degree of imprecision. In contrast, straight edge and compass constructions are, at least in principle, exact.

What formal constructions do you remember doing when you were at school? Here are some that you might remember.

- Construct the mid-point of a given line segment.
- Construct the perpendicular bisector of a given line segment.
- Construct the bisector of a given angle.
- Construct the perpendicular to a given line segment from a given point not on the line.
- Construct the perpendicular to a given line segment at a given point on the line.

You will need a **pencil**, **straight edge** and **compasses**. In your personal file, draw as many as you can of the constructions above.

As you work, think about what you are doing, and *why* – for example, by considering as you draw why any arcs and points are marked as they are.

When you have finished, check what you have done by looking at the supplement of examples, Framework section 4, page 221, focusing on the examples for Year 8.

2 Think about how each of the constructions on Framework page 221 links to the properties of a rhombus. Here are some possibilities.

- Diagram 1: the diagonals of a rhombus bisect at right angles.
- Diagram 2: the diagonals of a rhombus bisect the interior angles.

- Diagram 3: the diagonals of a rhombus intersect at right angles.
- Diagram 4: the diagonals of a rhombus intersect at right angles.

It is important to help pupils to make connections between different aspects of mathematics. Here, for example, there is an opportunity for pupils to appreciate how familiar properties of the rhombus can be applied to solving problems on construction.

3 Look at the references to constructions in the teaching programmes for Years 7, 8 and 9, Framework section 3, pages 7, 9, 11. Look also at the linked examples in the supplement of examples, section 4, page 220–223.

As you study these references and examples, consider these points.

- Where do objectives involving constructions appear?
- Are there any key objectives? If so, in which year?
- What examples are illustrated in the supplement? Would any of these be useful to incorporate into mathematics lessons for the classes that you teach? If so, make a note of them in your personal file.

4 Look at the sample unit 'Construction and loci (8S1)' in **Resource 6d, Sample teaching unit.** The department using this unit decided to teach loci before constructions to allow pupils to learn from the practical experience of generating sets of points and to make connections between this and their work on constructions.

Read the sample unit and match it to the objectives for 'Construction and loci' in the Year 8 teaching programme (Framework section 3, page 9).

Unit plans don't have to be presented like the example in Resource 6d. There are many different ways of recording them effectively, although it is likely that they all illustrate the same important features. Try to identify what these important features are, then check how they appear in the scheme of work for your own school.

5 Look at **Resource 6e, Key Stage 3 National Curriculum tests: questions on loci**. The questions are all at level 7.

What skills and knowledge would pupils need in order to answer the questions successfully?

For each test question, think about the kinds of informal or formal arguments that you would expect pupils to give if asked to justify their reasoning.

Part 5 Summary of Modules 5 and 6

1 Pupils can be aware of and use geometrical facts or properties that they have discovered intuitively from practical work before they can prove them analytically. The aim in Key Stage 3 is for pupils to use and develop their knowledge of shapes and space to support geometrical reasoning. Teach them to understand and use short chains of deductive reasoning and results about alternate and corresponding angles to reach a proof. Later, pupils should be able to explain why the angle sum of any quadrilateral is 360°, and to deduce formulae for the area of a parallelogram and of a triangle from the formula for the area of a rectangle. These chains of reasoning are essential steps towards the proofs that are introduced in Key Stage 4.

Geometry cannot be learned successfully solely as a series of logical results. Pupils also need opportunities to use instruments accurately, draw shapes and appreciate how they can link together or be dissected. It is vital to distinguish between the imprecision of constructions that involve protractors and rulers, and the 'exactness in principle' of standard constructions that use only compasses and a straight edge. Geometrical reasoning can show pupils why construction methods work – for example, the method to construct a perpendicular bisector of a line segment.

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2 Look back over the notes you have made during this module. Have you identified what you may need to consider and adopt in your planning and teaching of geometry?

Use **Resource 6f, Summary and further action on Module 6**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and points to discuss with your head of department.

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3 If you are interested in reading more about the teaching of geometry in secondary schools, read *Teaching and learning geometry 11–19*, a joint report from the Royal Society and the Joint Mathematical Council. This report reiterates the centrality of geometry to the mathematics curriculum and how important it is that this branch of the subject should not be neglected. You can download the report from http://www.royalsoc.ac.uk/downloaddoc.asp?id=1191.

If you have not already done so, you could also download and look at the Key Stage 3 Strategy's *Year 9 geometrical reasoning: mini-pack* from http://www.standards.dfes.gov.uk/keystage3/respub/ma_intery9geom.

Resource 6a 3-D visualisation activities

The first part of each of these activities should be carried out without any drawing.

1 SKELETON PYRAMID

Imagine a wire-framed skeleton of a square-based pyramid.

How many faces does the pyramid have?

How many edges are there? How many vertices?

Imagine locating the centre of each face.

Imagine each centre joined to the centres of its adjacent faces.

The lines joining these centres form the skeleton of another 3-D shape.

Write the name of this new shape in the box below.

The new shape is a ………………………………………………………………………..

2 SKELETON CUBE

Imagine a wire-framed skeleton of a cube.

How many faces does the cube have?

How many edges are there? How many vertices?

Imagine locating the centre of each face.

Imagine each centre joined to the centres of its adjacent faces.

The lines joining these centres form the skeleton of another 3-D shape.

Write the name of this new shape in the box below.

The new shape is a ………………………………………………………………………..

[continued on the next page]

In the box below, jot down some questions that you could ask pupils that would follow on from this visualisation activity.

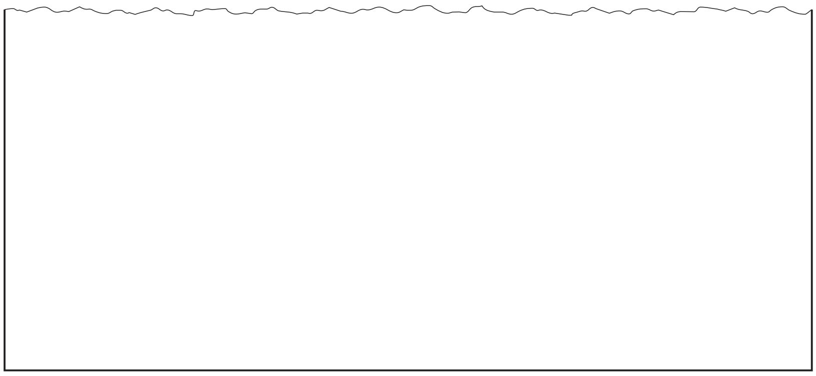
Resource 6b Moving a coin

For these activities you will need a blank sheet of A4 paper, about 30 cm of string, a 5p coin, a pencil, a ruler, a set square and compasses.

LOCUS 1

Place the 5p coin so that its centre is about 5 cm from a long edge of the paper.
Push the coin so that it is always the same distance from the long edge of the paper.
Use the string to check that the distance from the edge stays constant.

Make a freehand sketch of the path of the centre of the coin on the rectangle below.



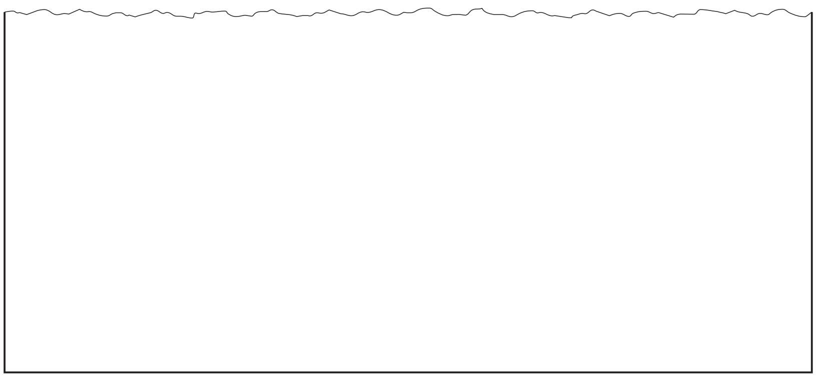
Write a description of the path of the coin.

The coin moves along a line that is:

LOCUS 2

Place the 5p coin so that its centre is an equal distance from a long edge and a short edge of the paper.
Push the coin so that its centre is always the same distance from the long and short edges.
Use the string to check that the distances remain the same.

Make a freehand sketch of the path of the coin on the rectangle below.



Write a description of the path of the coin.

The coin moves along a line that is:

LOCUS 3

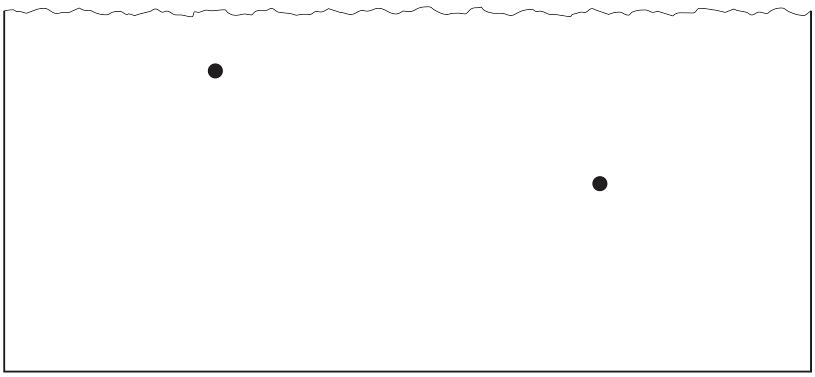
Mark two points on the A4 paper, about 8 cm apart.

Place the 5p coin so that it is an equal distance from each of the two points.

Push the coin so that it is always the same distance from the two points.

Use the string to check that the measurements are correct.

Make a freehand sketch of the path of the coin on the rectangle below.



Write a description of the path of the coin.

The coin moves along a line that is:

Resource 6c Loci problems

LOCUS 4

A dog is trained to walk around a rectangular flower bed in a garden so that it always remains 2 metres from the edge of the flower bed.
The flower bed is 2 metres wide by 3 metres long.

Draw the locus of points along which the dog walks. Think carefully about what happens at the corners.

Write a description of the locus using correct mathematical terms.

LOCUS 5

Two fixed points A and B are 6 cm apart.

Draw the locus of points P so that the total distance A to P plus P to B is always 10 cm. A ruler, or even a piece of string and pins at A and B, might help.

What will happen to this locus as A and B move closer together?

What will happen to this locus as A and B move further apart?

Construction and loci (8S1; 6 hours)

This Year 8 unit on Shape, space and measures follows on from the Year 7 unit 7S5, in which pupils constructed triangles using a ruler and protractor (SAS and ASA). In this unit, pupils are introduced to the notion of a locus through practical tasks (using people and/or counters on the OHP). This is developed into constructions, making links clear. Pupils should be encouraged to use their reasoning skills to draw out the properties of shapes (e.g. diagonals of a rhombus) and to do the four standard constructions.

Teaching objectives for the oral and mental starters

- Estimate and order acute, obtuse and reflex angles.
- Visualise, describe and sketch 2-D shapes and describe their properties, including symmetries.
- Visualise and describe the effects of reflections and translations on 2-D shapes.
- Visualise simple loci.
- Multiply and divide integers and decimals by 10, 100 and 1000 and explain the effect (converting between metric units in preparation for unit 8S2).
- Select some mental and oral number skills from previous work to reinforce learning.

Teaching objectives for the main activities

Simplification (Y7 objectives)	Core (mainly Y8 objectives)	Extension (Y9 objectives)
• Use a ruler and protractor to measure and draw lines to the nearest millimetre and angles, including reflex angles, to the nearest degree.	• Use straight edge and compasses to construct: – the mid-point and perpendicular bisector of a line segment; – the bisector of an angle; – the perpendicular from a point to a line; – the perpendicular from a point on a line.	
• Use a ruler and protractor to construct a triangle given two sides and the included angle (SAS) or two angles and the included side (ASA); explore these constructions using ICT.	• Construct a triangle given three sides (SSS). • Use ICT to explore these constructions. • Find simple loci, both by reasoning and by using ICT, to produce shapes and paths, e.g. an equilateral triangle. • Use correctly the vocabulary, notation and labelling conventions for lines, angles and shapes. • Present and interpret solutions, justifying inferences and explaining reasoning, using step-by-step deduction; give solutions to an appropriate degree of accuracy in the context of the problem.	• Use straight edge and compasses to construct a triangle, given right angle, hypotenuse and side (RHS). • Use ICT to explore constructions of triangles and other 2-D shapes. • Explain how to find, calculate and use the sums of the interior and exterior angles of regular polygons.

Activities and key teaching points

Oral and mental starters	Notes
Use questions that require pupils to order a range of angles (acute, obtuse, reflex).	
Use questions that require pupils to visualise and describe shapes and their properties, e.g. reflections, translations, rotations.	Ensure that some questions include 3-D shapes.
Use questions that require pupils to visualise and describe paths to generate loci.	See Framework examples page 225.
Practise conversion of metric units: mm to cm, cm to m, m to km, and vice versa.	
Practise mental addition and subtraction of decimals (two-digit numbers).	
Main activities	
Simple loci	
Use practical tasks to develop pupils' appreciation of locus. Emphasise the use of simple loci to produce shapes and paths.	The word 'locus' comes from the Latin word for 'place'. Locus may initially be thought of as the path traced out by a point as it moves under given conditions. It should then be defined as the set of all points that satisfy these given conditions.
Activity: Understanding locus as the path traced out by a point.	Framework examples pages 224–227.
Activity: Place counters on a table according to a given rule and determine the locus generated.	Framework examples pages 224–227.
Both these activities can be demonstrated effectively using counters on an OHP, using pupils as points or counters, or using dynamic geometry software.	Using a computer and large screen, ICT can be used to generate shapes and paths. For more able pupils, extend this work to include Year 9 examples.
Constructions	
Emphasise the use of compasses to construct all points at a fixed distance from a point. Use examples such as:	Construction of equilateral triangles may be appropriate here.
• Given a line AB, ask pupils to show the points that are 3 cm from A and 4 cm from B. Use this as an introduction to drawing triangles given three sides (SSS) using compasses.	Link to locus examples. See Framework examples page 221.
• Ask pupils to construct a rhombus using a	In Year 7 pupils constructed a

straight edge and compasses, given the length of a diagonal and the length of a side. Provide an opportunity for pupils to review the properties of the rhombus.	rhombus given the length of one side and one of the angles. Pupils should use their knowledge of the properties of the rhombus to explain why the construction works.
Activity: Use straight edge and compasses for constructions. Teach pupils to construct: • the perpendicular from a point to a line; • the perpendicular from a point on a line; • the midpoint and perpendicular bisector of a line segment; • the bisector of an angle. Give pupils a range of examples to use these constructions, e.g. from the textbook, worksheet and OHTs available in the department resource file.	Framework examples pages 220–223.
Constructions and loci	
Use ICT software (dynamic geometry and Logo) to generate shapes and paths.	Framework examples pages 224–227.
Activity: Use ICT to generate shapes and paths.	Framework examples pages 224–227.
Ask pupils to construct rectilinear shapes, regular polygons and equi-angular spirals. Ask pupils to explore the construction of regular polygons using Logo. Ask pupils to produce nets for regular tetrahedron and octahedron.	Year 9 objectives include the calculation of internal and external angles of regular polygons. More able pupils should generalise results while others continue to experiment.

Resources

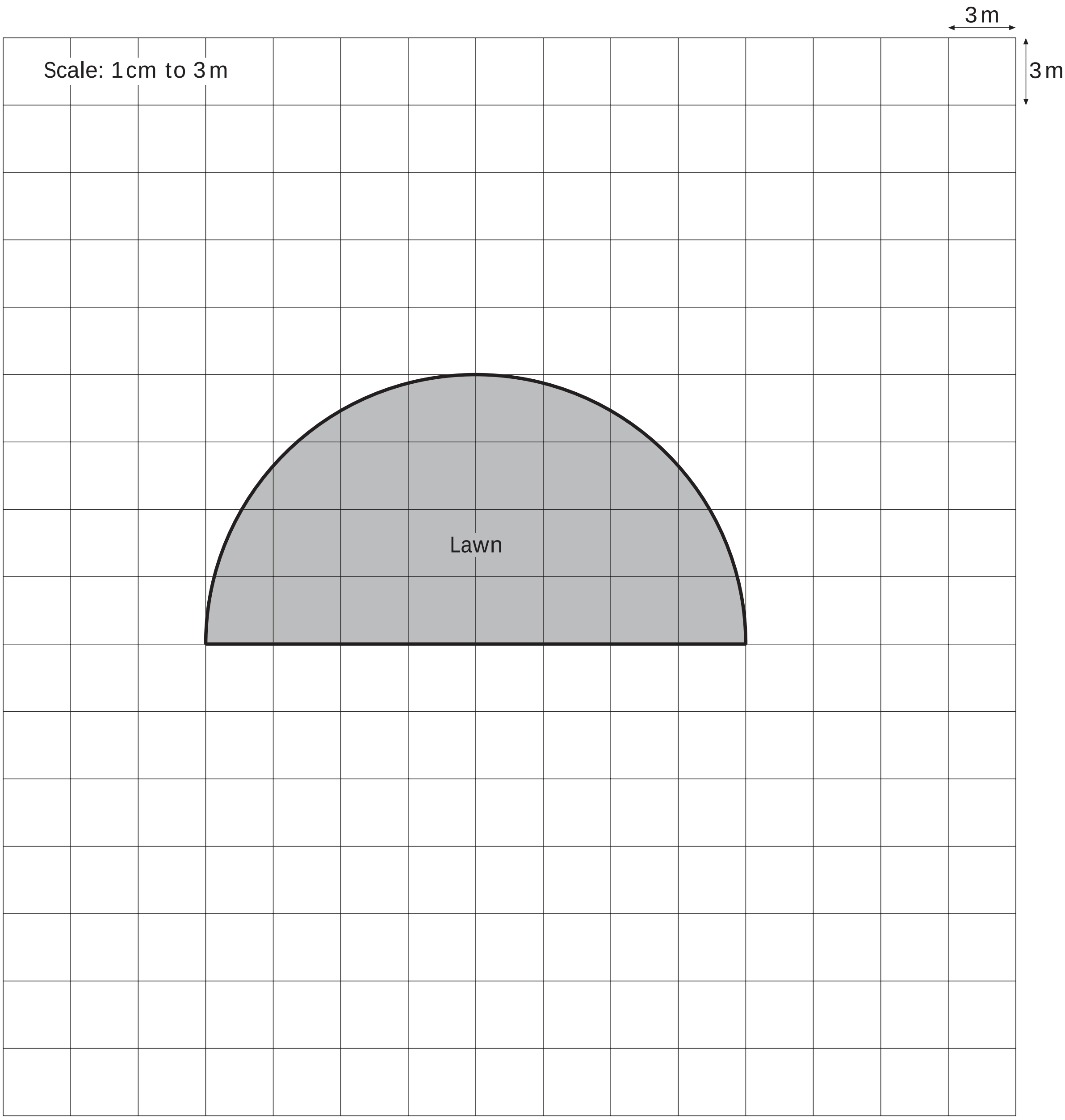
Counters, OHP, rulers, compasses, plain paper, ICT software (dynamic geometry, Logo)

Key vocabulary

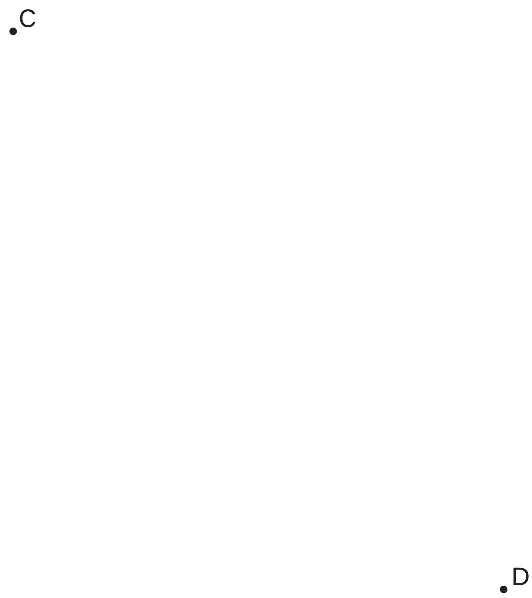
Bisect, bisector, perpendicular bisector, mid-point, line segment, equidistant, compasses, locus, loci

Resource 6e Key Stage 3 National Curriculum tests: questions on loci

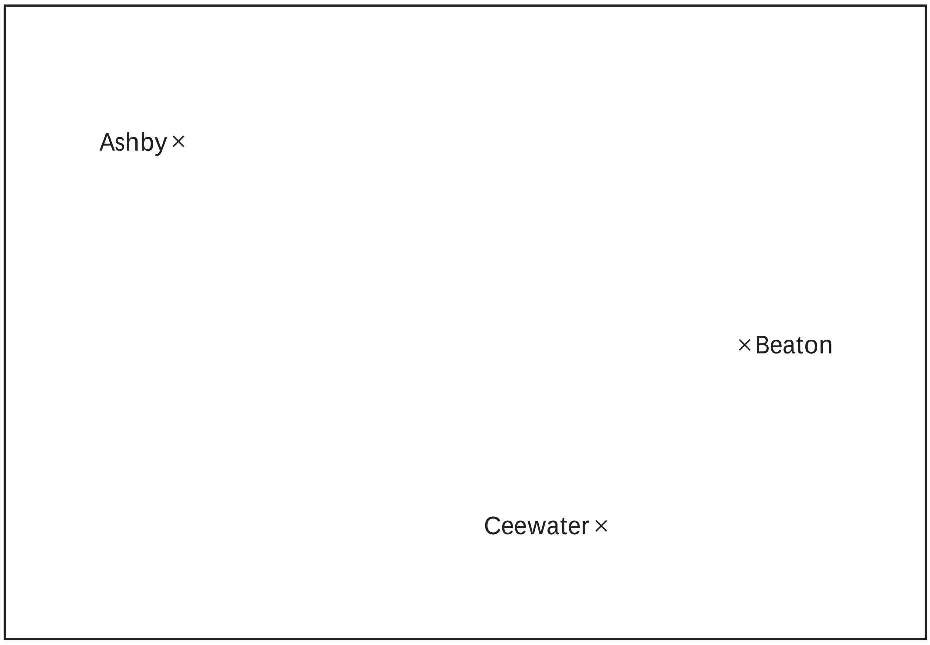
1 In the scale drawing, the shaded area represents a lawn.
There is a wire fence all around the lawn.
The shortest distance from the fence to the edge of the lawn is always 6m.
On the diagram, draw accurately the position of the fence.



2 Look at points C and D below.
Use a straight edge and compasses to draw the locus of all points that are the same distance from C as from D.
Leave in your construction lines.



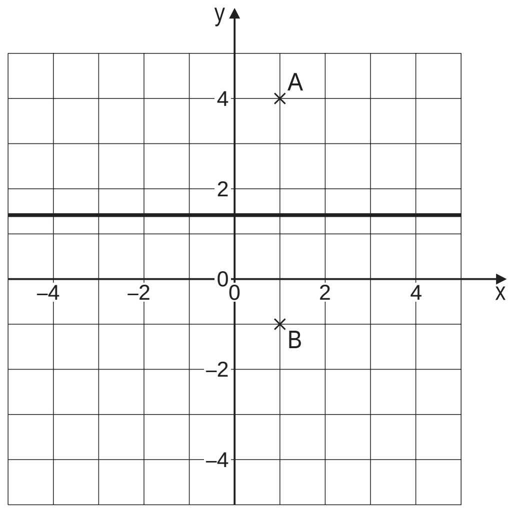
3 The plan shows the position of three towns, each marked with a ×.
The scale of the plan is 1 cm to 10 km.



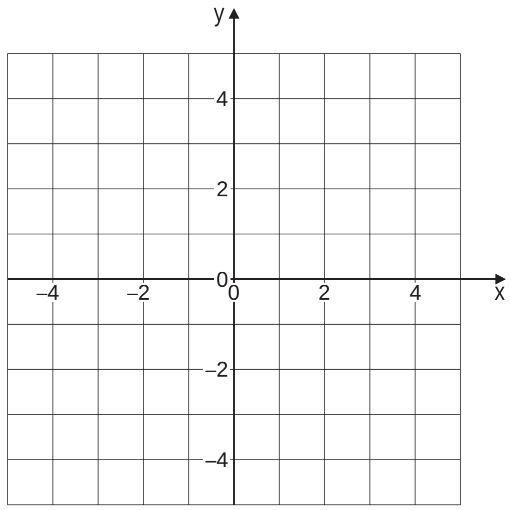
The towns need a new radio mast. The new radio mast must be:
nearer to Ashby than Ceewater,
and less than 45 km from Beaton.

Show on the plan the region where the new radio mast can be placed.
Leave in your construction lines.

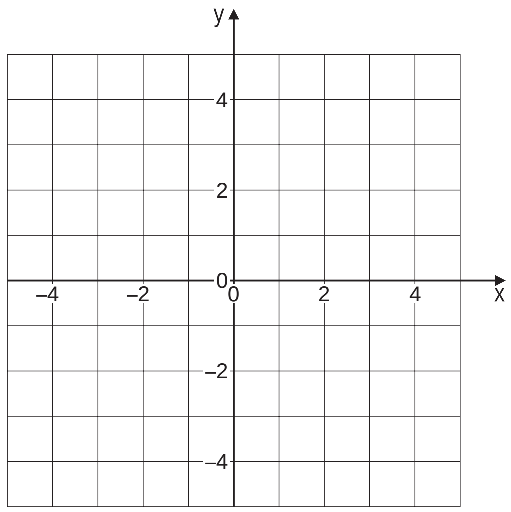
4 The diagram shows the locus of all points that are the same distance from A as from B. The locus is one straight line.



The locus of all points that are the same distance from (2, 2) and (–4, 2) is also one straight line. Draw this straight line.



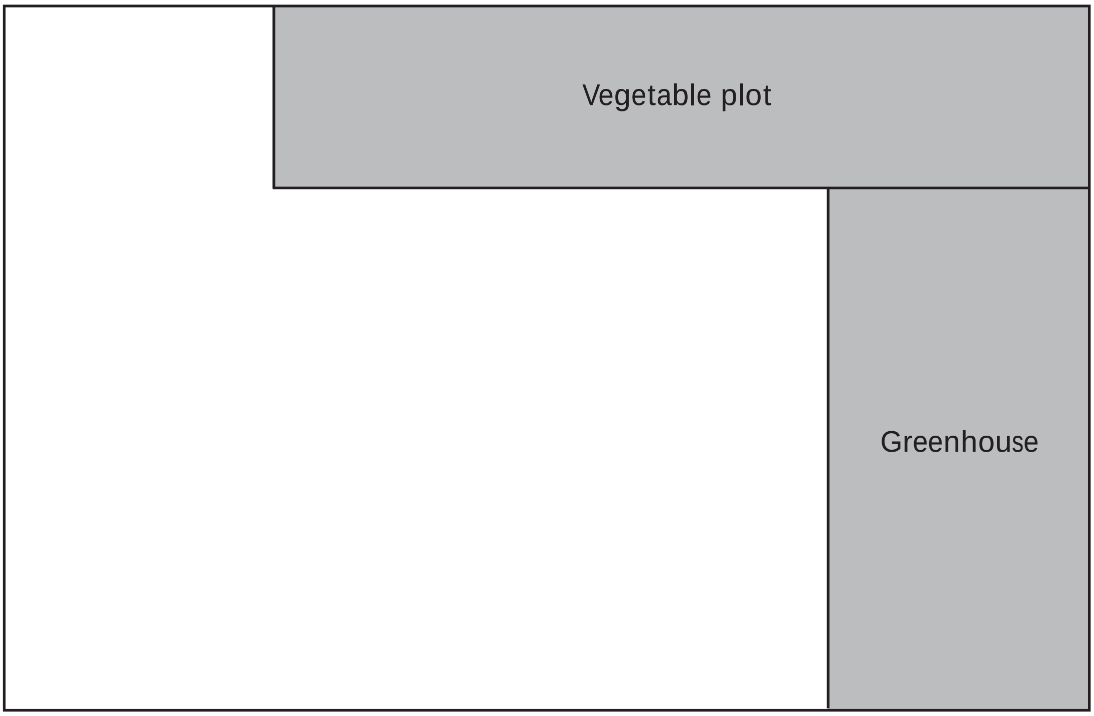
The locus of all points that are the same distance from the *x*-axis as they are from the *y*-axis is two straight lines. Draw both straight lines.



5 A gardener wants to plant a tree.
She wants it to be more than 8 m away from the vegetable plot.
She wants it to be more than 18 m away from the greenhouse.

The plan below shows part of the garden.
The scale is 1 cm to 4 m.

Show accurately on the plan the region of the garden where she can plant the tree.
Label this region R.



Resource 6f Summary and further action on Module 6

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching of geometry.

List two or three key points that you have learned.

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List two or three points to follow up in further study.

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List two or three modifications that you will make to your planning or teaching of geometry.

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List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

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MODULE

7

Ratio and proportion 1

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- discusses ratio and proportion as key mathematical ideas, with applications across many aspects of Key Stage 3 mathematics and in other subjects;
- analyses a lesson on ratio and proportion.

CONTENT

The module is in five parts.

1 Introduction
2 Fractions, ratio and proportion
3 Looking at a lesson on ratio and proportion
4 Proportion as a functional relationship
5 Summary

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- The *Framework for teaching mathematics: Years 7, 8 and 9*
- Video sequence 4, a Year 7 lesson on ratio and proportion, from the CD-ROM accompanying this module
- The resource sheets at the end of this module:
  - 7a Extract adapted from section 1 of the Framework
  - 7b Introducing ratio and proportion
  - 7c Walt's lesson plan
  - 7d Observing Walt's lesson
  - 7e Proportions and graphs
  - 7f Proportion as a functional relationship
  - 7g Summary and further action on Module 7

Desirable

- *What is a fraction?*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_fraction
- *Year 7 fractions and ratio: mini-pack*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_prop_rea
- *Year 8 multiplicative relationships: mini-pack*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_intery8multi
- *Year 9 proportional reasoning: mini-pack*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_y9_prop

STUDY TIME

Allow approximately 75 minutes, plus reading time for *What is a fraction?*

Part 1 Introduction

1 Before you study this module, aim to download and read *What is a fraction?* http://www.standards.dfes.gov.uk/keystage3/respub/ma_fraction.

The topic of ratio and proportion is a key aspect of the number curriculum in Key Stage 3. It addresses ideas that permeate all branches of mathematics and has many applications in other subjects.

In your personal file, jot down some applications of ratio and proportion which pupils are likely to meet at Key Stage 3, in mathematics and in other subjects.

Read **Resource 7a, Extract adapted from section 1 of the Framework**.

2 The topic of ratio and proportion is often regarded as one that gives rise to some difficulties in teaching and learning. Modules 7 and 8 consider:

- how the underlying ideas can be clarified;
- how links between different aspects can be drawn out.

Part 2 Fractions, ratio and proportion

1 Pupils first meet fractions as parts of whole numbers – particularly in relation to the operation of sharing or dividing into equal parts. In Years 5 and 6, pupils working at levels 4 and 5 meet the ratio aspect of fractions. This aspect features more prominently in Key Stage 3.

Read and do the activities in **Resource 7b, Introducing ratio and proportion**.

2 The proportional sets that can be generated from the set of integers 1 to 10 are:

$\frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5} = \frac{6}{6} = \frac{7}{7} = \frac{8}{8} = \frac{9}{9} = \frac{10}{10}$

$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10}$ and $\frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \frac{10}{5}$

$\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$ and $\frac{3}{1} = \frac{6}{2} = \frac{9}{3}$

$\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$ and $\frac{3}{2} = \frac{6}{4} = \frac{9}{6}$

$\frac{1}{4} = \frac{2}{8}$ and $\frac{4}{1} = \frac{8}{2}$

$\frac{3}{4} = \frac{6}{8}$ and $\frac{4}{3} = \frac{8}{6}$

$\frac{1}{5} = \frac{2}{10}$ and $\frac{5}{1} = \frac{10}{2}$

$\frac{2}{5} = \frac{4}{10}$ and $\frac{5}{2} = \frac{10}{4}$

$\frac{3}{5} = \frac{6}{10}$ and $\frac{5}{3} = \frac{10}{6}$

$\frac{4}{5} = \frac{8}{10}$ and $\frac{5}{4} = \frac{10}{8}$

3 In summary:

- **Ratio** is a way of comparing two or more quantities measured in the same units – the quantities may be separate entities or they may be different parts of a whole.
- The definition of **proportion** given here, 'an equality of ratios', is an essential mathematical term. Often we use the same word to refer to a fractional part of a whole (for example, the proportion of pupils in a class who are absent). This can sometimes be a source of confusion.

- If two ordered sets of numbers $\{a, b, c, \ldots\}$ and $\{q, r, s, \ldots\}$ are such that $a : q = b : r = c : s = \ldots$ (or — = — = — = …), then the two ordered sets are said to be in **direct proportion**.

Part 3 Looking at a lesson on ratio and proportion

1 In this part of the module, you will watch a lesson on ratio and proportion. Walt teaches the third of a series of lessons on this topic to a top set in Year 7. Study **Resource 7c, Walt's lesson plan**, before you watch the lesson.

---

2 Get ready to watch **Video sequence 4, a Year 7 lesson on ratio and proportion**. The teacher is Walt. The lesson has three distinct parts. Watch each part of the lesson separately: first the starter, then the main part of the lesson, and then the final plenary. After each part, consider and make notes on the questions on **Resource 7d, Observing Walt's lesson**.

The whole video sequence lasts about 13 minutes: 4 minutes for the starter, 4 minutes for the main activity and 5 minutes for the plenary.

When you have finished watching, spend a few minutes completing the notes you have made on Resource 7d.

---

3 Did you notice these features of Walt's lesson?

Starter

- Walt makes effective use of the OHP to review work from the previous lesson.
- Key vocabulary is displayed and Walt uses the terms associated with ratio and proportion very precisely.
- The sequence with the counting stick practises the skill of generating equivalent ratios, a skill which will be used in the main part of the lesson.

Main activity

- Pupils find their own ways of adapting the approach of finding equivalent ratios to particular examples.
- Walt provides a helpful model for recording the steps of the calculation, a model the pupils readily take on board.

Plenary

- Walt relates the same mathematical ideas to a completely different context.
- The context is introduced in an imaginative way and hints at aspects of work which pupils will meet in the future, for example, scale and enlargement.

Part 4 Proportion as a functional relationship

1 Because proportions are a particular type of functional relationship, they will occur in algebra as part of work on functions and graphs. Read **Resource 7e, Proportions and graphs**.

---

2 Think about and make notes on your answers to the questions on **Resource 7f, Proportion as a functional relationship**.

3 Study the yearly teaching programmes and supplements of examples on ratio and proportion in the Framework, sections 3 and 4. Relevant references for the supplement of examples are in the yearly teaching programmes.

Focus on the progression across Years 7, 8 and 9. Look for ways in which explicit links between different objectives might be made.

As you study, make notes in your personal file on relevant points in your school's scheme of work when you can strengthen pupils' awareness of the links. Note also any points that you want to discuss later with your head of department.

Part 5 Summary

1
- A proportion is an equality of ratios.
- Equivalent fractions (or ratios) are formed from proportional sets of numbers, in which there is a constant multiplier relating numerators to denominators.
- Where two or more sets of numbers represented by the variables (x, y) are such that $\frac{y}{x} = m$, or $y = mx$, where m is a constant, the numbers are in direct proportion, and m is called the constant of proportionality.
- Where x and y are variable numbers, m describes the rate of change of y with respect to x, that is, y changes by m for every 1 of x. The phrases 'for every', 'in every' and 'to every', or simply 'per', are all used when describing a constant rate – for example, miles per hour.
- m is sometimes thought of as a scale factor.
- The graph of $y = mx$ consists of points lying in a straight line through the origin. Graphically, the rate of change is represented by 'rise ÷ step', and m is the gradient of the graph.

2 Look back over the notes you have made during this module. Have you identified what you may need to consider and adopt in your planning and teaching of ratio, proportion and proportional reasoning?

Use **Resource 7g, Summary and further action on Module 7**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and any points to discuss with your head of department.

3 You may find it useful to download and study the Key Stage 3 mini-packs that support the teaching of ratio and proportion:

- *Year 7 fractions and ratio: mini-pack*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_prop_rea
- *Year 8 multiplicative relationships: mini-pack*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_intery8multi
- *Year 9 proportional reasoning: mini-pack*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_y9_prop

Resource 7a Extract adapted from section 1 of the Framework

This short article is based on pages 12 and 13 of section 1 of the Framework.

Proportional reasoning in Key Stage 3

Throughout Key Stage 3 pupils will extend their understanding of the number system to positive and negative numbers and, in particular, to fractions and their representations as terminating or recurring decimals.

Fractions, decimals, percentages, ratio and proportion are different ways of expressing related ideas and relationships. The connections start to be established in Key Stage 2, particularly the equivalence between fractions, decimals and percentages. The ideas of ratio and proportion, and the relationship between them, should be a strong feature of work in Key Stage 3. By the end of the key stage, pupils should be able to solve problems involving fractions, decimals, percentages, ratio and proportion, and their interconnections.

After calculation, the application of proportional reasoning is the most important aspect of elementary number. Proportionality underlies key aspects of number, algebra, shape, space and measures, and handling data. It is also central in applications of mathematics in subjects such as science, technology, geography and art. The study of proportion begins in Key Stage 2 but it is in Key Stage 3 where secure foundations need to be established.

Problems involving proportion are often solved by informal methods, particularly when the numbers involved are easy to deal with mentally. But it is important to teach methods that can be applied generally. For example, the unitary method is useful for solving problems involving proportion, and multiplicative methods involving fractions or decimals are useful for solving percentage problems.

Teaching proportional reasoning

When you are teaching proportional reasoning:

- emphasise the language and notation of ratio and proportion, and the links to fractions, decimals and percentages;
- teach pupils specific methods for solving proportion problems so that they do not remain dependent on informal approaches;
- help pupils to understand what they are calculating: for example, a distance divided by a time gives a speed – an example of a rate; but a distance divided by another distance gives a scale factor or multiplier – a dimensionless number;
- make explicit links between ideas of proportionality in number, algebra, shape, space and measures, and handling data.

Links with other mathematical topics

In algebra, direct proportion is viewed as a linear relationship of the form $y = mx$. The graphical representation of this equation helps pupils to visualise ideas such as rate of change and gradient. The algebraic representation of a proportion (e.g. $a : b = c : d$ or $a/b = c/d$) underpins a general method for solving problems.

In shape, space and measures, proportionality arises when enlargement by different scale factors is considered. Scaling has a wide range of applications, for example, in maps, plans and scale drawings. Similar figures have sides or dimensions that are in proportion. Recognition of the similarity of all circles leads to an understanding that the circumference is directly proportional to the diameter, while awareness of the similarity of triangles with the same angles leads to an understanding of trigonometry.

In statistics, proportions are often calculated when data are interpreted and inferences drawn. Proportions are also used when probabilities are estimated or calculated based on outcomes that, in theory, are equally likely.

Applications of ratio and proportion

Some of the many possible applications of ratio and proportion that pupils are likely to meet or could be introduced to in Key Stage 3, either in mathematics or in other subjects, are as follows.

- The cost of vegetables in the market, at a fixed price per kilogram, is proportional to the weight of vegetables purchased.
- A percentage increase or decrease is a proportion of the original amount or quantity.
- The distance between two points on a map drawn to scale is proportional to the actual distance on the ground.
- Athletes often try to improve their power-to-weight ratio in training.
- Bicycles have gears in different ratios to suit different conditions.
- The ratio of pupils to teachers in secondary schools is less than in primary schools.
- The scale factor of an enlargement is the ratio of corresponding lengths on the object and its image.
- When jogging at a steady 7 mph, the distance travelled is proportional to the time.
- The current I flowing in a circuit of fixed resistance R is proportional to the voltage applied V; that is, $V = IR$.
- The acceleration a of a mass m is proportional to the applied force F; that is, $F = ma$.
- On a hill of uniform slope, the height gained is proportional to the distance travelled along the slope. The gradient of the slope is the ratio of the increase in height to the horizontal distance.

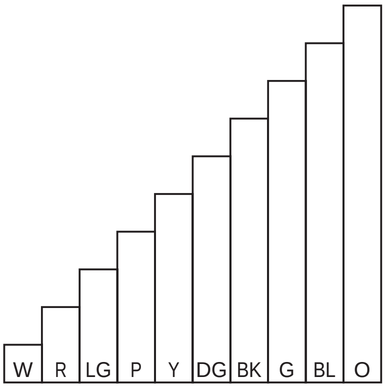
Resource 7b Introducing ratio and proportion

RATIO

Ratio is a way of comparing two numbers or quantities, by expressing how many times one number can be divided by the other.

The view of many pupils is that ratio is used only to compare parts of a whole. This is only one aspect of ratio. Ratio is not necessarily a part/part relationship.

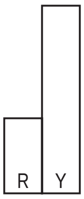
Assume that we have a set of coloured strips.



We will adopt shorthand for the lengths of the strips, using the symbol l_Y or l_R, for example, to represent the number of units of length of a strip:

1 unit	white (l_W)
2 units	red (l_R)
3 units	light green (l_{LG})
4 units	pink (l_P)
5 units	yellow (l_Y)
6 units	dark green (l_{DG})
7 units	black (l_{BK})
8 units	grey (l_G)
9 units	blue (l_{BL})
10 units	orange (l_O)

If we place two strips side by side, for example the red (2 unit) strip and the yellow (5 unit) strip, they would look like this:



We can compare the lengths of the two strips by finding a *difference*, or alternatively by finding a *ratio*.

ratio of $_{Y}$ to $_{R} = 5 : 2$

$$\frac{_Y}{_R} = \frac{5}{2}$$

$$_Y = \tfrac{5}{2} \times {_R} \quad (\text{or } 2\tfrac{1}{2} \times {_R})$$

ratio of $_{R}$ to $_{Y} = 2 : 5$

$$\frac{_R}{_Y} = \frac{2}{5}$$

$$_R = \tfrac{2}{5} \times {_Y}$$

Pick two different strips of your own choice and draw them below. Write out the ratios as in the example above.

Repeat with another two strips.

Another way of using coloured strips to illustrate ratio is to line up multiples ('trains') of each strip so that two coloured 'trains' are the same length, and then count the number of strips in each equal-length train. (Compare this with finding a common multiple.)

For example, the length of five red strips equals the length of two yellow strips, so the ratio $l_R : l_Y$ equals 2 : 5. (It is important to get the order correct; comparing the lengths of the two initial strips helps.)

When pupils are working with ratios they need to understand that:

- **Ratio involves division of one number by another.**
  - Sometimes the result will be a whole number.
  - Sometimes the result has to be left as a fraction or mixed number (or converted to an equivalent decimal or percentage form).
- **There are several ways of expressing a ratio:**
  - using colon notation;
  - using the fraction or division line (reading '/' as '÷');
  - using multiplication (reading '×' as 'of').
- **The inverse of the ratio *a* : *b* is *b* : *a*.**

  The associated fractions are — and its reciprocal — .

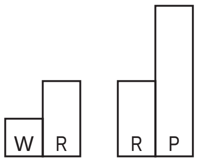
Other important points about the teaching of ratio and proportion are:

- At an appropriate stage, pupils should meet equivalent decimal and percentage forms of expressions, such as:
  $\frac{2}{5}$ = 0.4, which is equivalent to 40%;
  $\frac{5}{2}$ = 2.5, which is equivalent to 250%.
- Pupils' understanding is greatly assisted if they discuss different ways of expressing a ratio, rather than being introduced to different notations on completely separate occasions. This helps them to link ideas, use different forms and recognise equivalent expressions.
- The notion of an inverse operation is a key one in mathematics. In Key Stages 1 and 2, pupils are introduced to an operation and its inverse together – for example,

addition with subtraction and multiplication with division. Understanding inverse ratios will help pupils later – for example, to calculate the original amount given the amount after a specified percentage change.

PROPORTION

A **proportion** is an equality of ratios.



For these two pairs of coloured strips, the ratios are equal. What other pairs of strips have the same ratio?

Working with the coloured strips, the complete proportional set for the ratios 1 : 2 and 2 : 1 is:

$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10}$ and $\frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \frac{10}{5}$

Record all the other proportional sets you could make with the coloured strips.

Resource 7c Walt's lesson plan

Ratio and proportion: Year 7

ORAL AND MENTAL

Objectives

- Consolidate rapid recall of multiplication facts.
- Calculate mentally using ratio.

Activity

Remind class of concept of ratio, drawing on what they have done in previous two lessons.

Use counting stick to get pupils to follow the 4× and 10× multiplication tables simultaneously.

MAIN ACTIVITY

Objective

o Solve simple problems involving ratio.

Key vocabulary

Ratio, compare, 'for every', equivalent, lowest terms

Introduction

Remind class of previous activities involving ratio. Introduce exchange rates, leading to equivalent ratios and reducing to lowest terms. Ensure that pupils are clear on when they are using the same units. Use table from previous work on exchange rates to generate equivalent ratios.

Activity

Worksheet on ratio.

PLENARY

Show class picture by de Chirico and get pupils to spot the importance of shadows. Show shadow picture and discuss whether the shadows were observed at same time of day. Ask 'How can we tell?' 'What has ratio got to do with it?'

Homework

None

Resource 7d Observing Walt's lesson

Watch Walt working with his Year 7 top set. The lesson has three distinct parts: a starter, the main activities, and a final plenary.

Watch each part of the lesson separately. First, watch the starter, then consider and make notes on the questions below.

Oral and mental starter
• How did Walt make use of the OHP?   • What consideration did Walt give to the use of mathematical vocabulary?   • How did Walt make use of a counting stick?

Now watch the extract from the main part of the lesson. This shows Walt in discussion with two pairs of pupils who are working on proportion problems. When you have watched the extract, consider and make notes on the questions below.

Main activity
• How did pupils approach the ratio problems?   • What did the pupils record?

Now watch the final plenary, then consider and make notes on the questions below.

The plenary

- What strategies did Walt use to reinforce pupils' understanding of ratio and proportion?

Now that you have watched the whole lesson, think about it as a whole.

Reflections on the whole lesson

- Think about Walt's use of resources. Could you make use of similar resources with any of the classes that you teach? If so, make a note of them here.

- Were there any points about Walt's lesson that you would like to discuss with your head of department? If so, make a note of them here.

Resource 7e Proportions and graphs

Consider the set of fractions (or ratios) equivalent to 5:

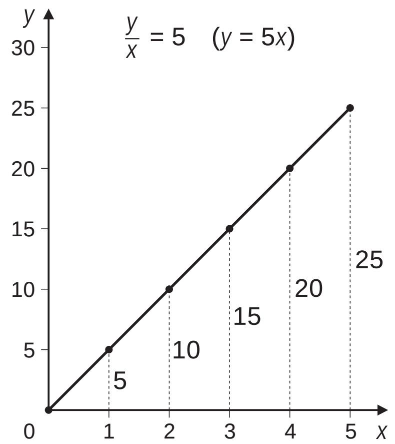
$$\frac{5}{1} = \frac{10}{2} = \frac{15}{3} = \frac{20}{4} = \frac{25}{5} = \ldots$$

The denominators {1, 2, 3, 4, 5, …} and numerators {5, 10, 15, 20, 25, …} are called **proportional sets** and the pairs of corresponding numbers (1, 5), (2, 10), (3, 15), (4, 20), (5, 25), … are said to be **in proportion**.

The relationship between these numbers can be illustrated graphically. Representing the set of denominators {1, 2, 3, 4, 5, …} by variable x and the set of numerators {5, 10, 15, 20, 25, …} by variable y, then:

$$\frac{y}{x} = 5 \quad (\text{or } y = 5x)$$

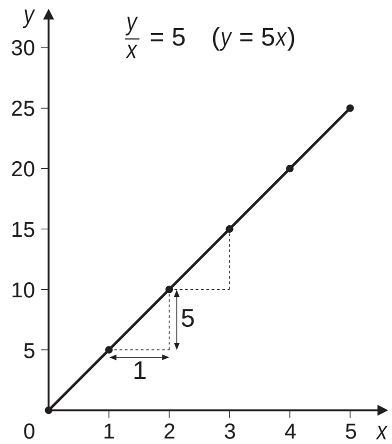
x	1	2	3	4	5	…
y	5	10	15	20	25	…



The graph illustrates the equal ratios $\frac{5}{1}, \frac{10}{2}, \frac{15}{3}, \cdots$, represented by the sides of right-angled triangles, enlarging from the origin. This visualisation of a proportion links with enlargement and similarity and with later applications in trigonometry, which will be considered in Module 8.

It is valuable for pupils to see the relationship $y = 5x$ expressed as $\frac{y}{x} = 5$, since this draws attention to the equal ratios.

Another aspect to consider is how y changes with x. From the table of values above the graph, we can see that y increases by 5 for every 1 of x. By drawing a staircase pattern of triangles, as shown below, we can see that rises of 5 in y correspond to steps of 1 in x. This is an example of a constant **rate of change**.



Pupils should notice that the graphs of all proportions (equal ratios) are straight lines through the origin, with constant rate of change.

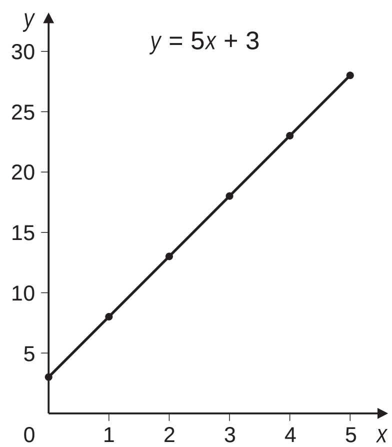
In its general form, the relationship $y = 5x$ is $y = mx$, where m is a constant. Functions of the form $y = mx$ are proportions, where m, which measures the **rate of change** or **gradient** of the graph, is called the **constant of proportionality**. (This type of relationship would be called **direct proportion** when it is necessary to distinguish it from **inverse proportion.**)

Depending on the quantities represented, the constant m may be thought of as a scale factor or multiplier (e.g. when dividing one distance or length by another) or as a rate, (e.g. when dividing distance by time, as in miles per hour).

Proportions are not the only relationships that involve a constant rate of change. Consider the function $y = 5x + 3$, and complete the following table of values:

x	1	2	3	4	5
y					
$\frac{y}{x}$					

The rate of change is still 5, but the ratios of successive pairs of values are not constant.



The relationship $y = 5x + 3$ is a **linear relationship**, since the graph is a straight line, but it does not go through the origin. The **general linear function** is of the form $y = mx + c$, where m is the rate of change of y with x. The constant m is a measure of the **gradient** of the graph and c is the **intercept** with the y-axis.

Resource 7f Proportion as a functional relationship

Consider the questions below and make notes on your responses.

What are the benefits to pupils if they are given opportunities to make the connections considered in this module?

How might an awareness of the connections help pupils when they are dealing with practical applications, whether in mathematics or other subjects?

Resource 7g Summary and further action on Module 7

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching of ratio, proportion and proportional reasoning.

List two or three key points that you have learned.

-
-
-

List two or three points to follow up in further study.

-
-
-

List two or three modifications that you will make to your planning or teaching of ratio, proportion and proportional reasoning.

-
-
-

List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

-
-
-

MODULE

8

Ratio and proportion 2

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- considers methods for solving problems involving ratio and proportion;
- discusses how mathematical problems and methods can be simplified or made more challenging to meet the needs of different pupils;
- highlights links between ratio and proportion and enlargement and similarity.

CONTENT

The module is in four parts.

1 Introduction

2 Solving problems on ratio and proportion

3 Enlargement and similarity

4 Summary

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- The *Framework for teaching mathematics: Years 7, 8 and 9*
- Video sequence 5, Year 6 pupils discussing test questions, from the DVD accompanying this module, and a DVD player
- A4 plain paper, scissors and glue, and a ruler
- A calculator
- The resource sheets at the end of this module:
  - 8a Key Stage 2 test questions involving ratio and proportion
  - 8b Year 6 pupils talking about test questions
  - 8c Different ways of solving a proportion problem
  - 8d Circle and diameter
  - 8e Metric paper sizes
  - 8f Summary and further action on Module 8

Desirable

- *Year 8 multiplicative relationships: mini-pack*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_intery8multi
- *Year 9 proportional reasoning: mini-pack*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_y9_prop

STUDY TIME

Allow approximately 90 minutes.

Part 1 Introduction

1 Modules 7 and 8 consider:

- how the ideas underlying ratio and proportion can be clarified;
- how links can be made between different mathematical topics that draw on ideas of ratio and proportion.

This module assumes that you have already studied Module 7.

---

2 The focus in Part 2 of this module is methods for solving problems on ratio and proportion. Usually the problem reduces to an equation of the form

$a : b = c : d$

where three of the numbers are known and the fourth has to be found. However, problems can appear in many different guises and several approaches are possible, some more sophisticated than others.

The focus for Part 3 of this module is enlargement.

As you consider approaches to ratio and proportion problems, there is a chance for you to reflect on:

- how the same problem might be approached by different pupils, depending on their level of understanding;
- how a problem can be simplified or made more challenging to suit the needs of different pupils;
- how pupils can move from informal to more formal methods.

Part 2 Solving problems on ratio and proportion

1 Look at **Resource 8a, Key Stage 2 test questions involving ratio and proportion**. The questions are from the May 2000 tests and pre-date the use of the euro in France.

- Test A (non-calculator) Question 15
- Test B (calculator) Question 21
- Test C (calculator) Question 1

Consider each question in turn. Think about and make a note of the possible alternative approaches and methods that could be used with each problem. Bear in mind whether or not the pupils will have had access to a calculator.

---

2 You now have an opportunity to watch responses from higher-attaining Year 6 pupils to the three questions you have just considered.

The pupils were interviewed in July, having completed the tests some weeks earlier in May 2000. Although the children are not a representative sample of pupils about to enter Key Stage 3, their responses illustrate issues that need to be addressed when teaching ratio and proportion in Years 7 to 9.

Get ready to watch **Video sequence 5, Year 6 pupils discussing test questions**.

Watch the video sequence for the first test question, then pause the video. Consider and make notes on the question on **Resource 8b, Year 6 pupils talking about test questions**. Then continue with the second and third questions similarly.

The whole video sequence lasts about 9 minutes: 3 minutes for the first question, 4 minutes for the second question and 2 minutes for the third question.

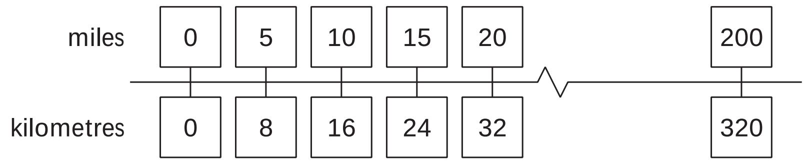
When you have finished watching, spend a few minutes completing the notes you have made on Resource 8b.

---

3 Read **Resource 8c, Different ways of solving a proportion problem**.

When pupils are working on ratio and proportion problems, teach them to do the following as a matter of routine.

- Consider whether the answer will be bigger or smaller than a given quantity.
  This is often sufficient to overcome common mistakes (e.g. in the Calais to Paris example, will the number of miles be more or less than 320?).
- Look at the numbers themselves.
  This will often help to identify the most efficient method of solution for a given problem.
- Set out their solution clearly.
  Often a picture or diagram can provide a powerful clue to the solution, as in the problem about the distance from Calais to Paris.



Notice how this diagram links to the use of the counting stick that Walt used in the lesson you watched in Module 7.

Whatever method they use, pupils need to understand and be able to express what they are doing at each stage of solving the problem.

Make sure that pupils think carefully about how to start a problem. Get them to compare different methods. Informal scaling or the four-cell method allows them to work within their understanding, provided that the numbers are amenable. Scaling, unitary and algebraic methods are more direct and work for any numbers.

Part 3 Enlargement and similarity

1 We will now consider the spatial aspect of proportion. This aspect of proportion is important; many pupils will develop their ability to reason through a proportional problem given in a spatial context.

Drawing out the links within mathematics allows pupils to develop a better understanding of concepts and moves away from the notion of isolated skills.

---

2 Consider one example of how links can be explored. Try the activities on **Resource 8d, Circle and diameter.**

---

3 Pupils who work on a visualisation task like the one on Resource 8d are likely to develop an *intuitive* awareness of the mathematical similarity of all circles and of the approximate relationship between circumference and diameter.

The visualisation task has an advantage over a practical, experimental approach. Measuring different circular objects, plotting a graph and estimating the relationship from the graph then requires pupils to observe a plausible pattern in experimental data. Here the result is apparent immediately. It can be related instantly to the circle and its circumference without the distraction of numbers and rounding.

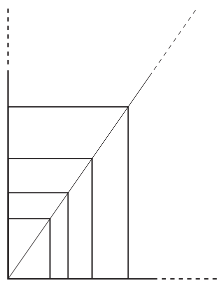
---

4 Now try the activity on **Resource 8e, Metric paper sizes**. You will need some sheets of A4 plain paper, scissors and glue, and a ruler.

---

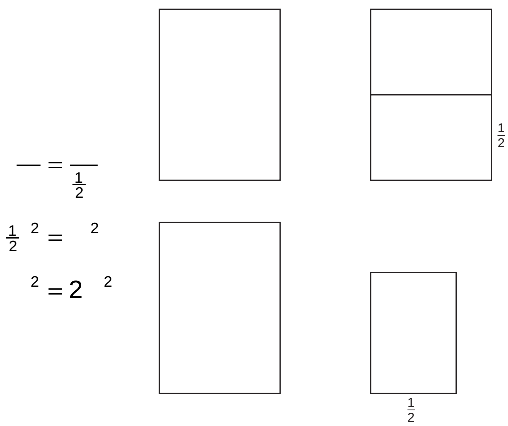
5 Compare your answers to the three problems on Resource 8e with the notes below.

Problem 1



Problem 2

Let *l* = length of paper and *w* = width of paper.



Since the area scale factor is $\frac{1}{2}$, the scale factor for the lengths must be $\frac{1}{\sqrt{2}}$.

Problem 3

Using the results of the previous two problems, together with fact that A4 will be $\frac{1}{16}$ of the area of A0, the dimensions of A4 are found to be 210 mm by 297 mm.

---

6 Like the work with strips of card in Module 7, diagrammatic work on enlargement of shapes provides a visual image for ratio and proportion. Diagrams can help pupils to understand scaling methods and the application of multipliers to solving problems (including inverse operations involving fractions, decimals and percentages).

It is worth noting the following points about the metric paper sizes problem.

- If pupils work on the problem of metric paper sizes, it is best if they start by investigating various rectangles of paper and confirm that folding them in half will not necessarily produce a mathematically similar shape. (Investigating old greetings cards may be the easiest way to do this.) This will help them to have a better appreciation of the uniqueness of the metric paper proportions.
- The metric paper proportion can be expressed in two ways:
  - as the ratio of length to width:

    $\frac{A0}{A0} = \frac{A1}{A1} = \ldots$
  - as a scaling (enlargement) from one size to the next:

    $\frac{A0}{A1} = \frac{A0}{A1}$ and so on

If you get an opportunity to do so, observe the factors of enlargement shown on a photocopier to enlarge from one paper size to the next. These are usually given as percentages.

---

7 Now study the supplement of examples, Framework section 4, pages 213–217. Look in particular at the Year 8 examples on enlargement and similarity, including the section on scale drawing.

To what extent do these examples correspond with your experiences as a teacher of mathematics? If there are differences, make a note of them in your points to discuss later with your head of department.

Think about how you could make the teaching of enlargement and similarity more explicit in your planning and teaching of Key Stage 3 lessons. Are there any examples in the Framework that would be useful to explore with the classes that you teach? Jot down your ideas in your personal file.

Part 4 Summary

1 Proportional reasoning is a key aspect of mathematics in Key Stage 3. It is important to:

- develop pupils' understanding of ratio, as distinct from fractions as part of a whole;
- ensure that pupils develop a range of effective methods for solving problems on proportion and are not reliant on just informal approaches;
- make explicit how proportional reasoning is used in different strands in mathematics and in other subjects, such as science, technology and geography: for example, there are direct links between enlargement and the graphical representation of functions and the measurement of gradients of lines, and also with trigonometry.

---

2 Look back over the notes you have made during this module. Have you identified all the factors that you want to consider and adopt in your planning and teaching of ratio and proportion?

Use **Resource 8f, Summary and further action on Module 8**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and any points to discuss with your head of department.

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3 If you have not already done so, you may find it useful to download and study the mini-packs on *Multiplicative relationships* and *Proportional reasoning*:

- *Year 8 multiplicative relationships: mini-pack*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_intery8multi
- *Year 9 proportional reasoning: mini-pack*
  http://www.standards.dfes.gov.uk/keystage3/respub/ma_y9_prop

Resource 8a Key Stage 2 test questions involving ratio and proportion

Consider each of these questions in turn. Think about and make a note of the possible alternative approaches and methods that could be made to each problem. Bear in mind whether or not the pupils will have had access to a calculator.

Test A Levels 3–5 Question 15 (calculator not allowed)

Peanuts cost 60p for 100 grams.
What is the cost of 350 grams of peanuts?

Possible approaches or methods to use for this problem:

Raisins cost 80p for 100 grams.
Jack pays £2 for a bag of raisins.
How many grams of raisins does he get?

Possible approaches or methods to use for this problem:

Test B Levels 3–5 Question 21 (calculator allowed)

A map shows that the distance from Calais to Paris is 320 kilometres.
5 miles is approximately 8 kilometres.
Use these facts to calculate the approximate distance in miles from Calais to Paris.

Possible approaches or methods to use for this problem:

Samira bought a present in France.
She paid 44.85 French francs for it.
9.75 French francs equal £1.
What was the cost of the present in pounds and pence?

Possible approaches or methods to use for this problem:

Test C Level 6 Question 1 (calculator allowed)

Shortcrust pastry is made using flour, margarine and lard.
The flour, margarine and lard are mixed in the ratio 8 : 3 : 2 by weight.

How many grams of margarine and lard are needed to mix with 200 grams of flour?

Possible approaches or methods to use for this problem:

Resource 8b Year 6 pupils talking about test questions

Watch the video sequence for each question separately. Then consider and make notes on the questions below.

1 Test A (non-calculator) Question 15

The sequence lasts 3 minutes.

How do the pupils' methods compare with those that you considered when you were studying the questions?

Where the pupils made errors, what would you do help them?

2 Test B (calculator) Question 21

The sequence lasts 4 minutes.

How do the pupils' methods compare with those that you considered when you were studying the questions?

Where the pupils made errors, what would you do help them?

3 Test C (calculator) Question 1

The sequence lasts 2 minutes.

How do the pupils' methods compare with those that you considered when you were studying the questions?

Where the pupils made errors, what would you do help them?

After watching the video

After you have finished watching the video, consider these questions.

How would the approaches you have seen in the video differ from what you would expect of pupils in Years 7, 8 and 9?

Are there any modifications that you want to make to your planning or teaching as a result of seeing and reflecting on the video extracts? If so, make some notes here on what you want to do.

Resource 8c Different ways of solving a proportion problem

There are several different ways of solving a proportion problem. The methods commonly used by pupils in the later stages of Key Stage 2 and in Key Stage 3 include:

- an informal scaling method;
- finding the scale factor;
- the unitary method;
- an algebraic method.

Informal scaling methods

Informal scaling methods are quite commonly seen, particularly as a mental or informal written approach. For example:

5 miles is about 8 km.
So 10 miles is about 16 km.
20 miles is about 32 km.
200 miles is about 320 km.

Such methods work well, provided that the scale factor is not difficult to find. Several of the Year 6 test examples in the video illustrate how the method can fail and the pupil is then forced to seek a more direct method.

Finding the scale factor

Finding the scale factor by division is more direct.

With the use of a calculator, this method is generally applicable whatever the numbers involved. However, pupils are not always able to express clearly and correctly what the answer to the division represents. For example:

320 km is $320 \div 8 = 40$ times as far as 8 km.
So 320 km is approximately $40 \times 5 = 200$ miles.

For example, one pupil divided 320 by 8 to get 40. On being asked what the 40 represented, she said '40 kilometres'. Another was able to say 'how many lots of 8 kilometres there are in 320 kilometres', recognising that the 40 is a dimensionless number (a multiplier or scale factor). This is a problem about understanding ratio. It is distinct from the other sense of a fraction as part of a whole, as in a question such as: 'I have travelled one eighth of the journey from Calais to Paris. How far have I travelled?'. This question requires the same calculation, but the answer is clearly a distance.

The four-cell method

This method provides some structure to the informal scaling and scale factor approaches. It involves creating a four-cell diagram and labelling the columns with the two variables. Three of the four cells are filled in, and the fourth has to be found.

miles	km	
5	8	
↓	↓	↓ $\times$ 40
?	320	

Since 8 has to be multiplied by 40 to make 320, the 5 must also be multiplied or 'scaled' by the factor 40, to give 200. The answer is therefore 200 miles.

A four-cell diagram provides a useful starting point for organising the information in a structured way to make the relationships between the numbers more obvious, and to identify the scale factor and calculation needed. The diagram can also act as a reminder that corresponding quantities in direct proportion problems need to be in the same units.

The unitary method

The unitary method is commonly taught in Key Stage 3. For example:

8 km is approximately 5 miles.
So 1 km is approximately $\frac{5}{8}$ *miles.*
So 320 km is approximately $\frac{5}{8} \times 320 = 200$ *miles.*

This method too is generally applicable. Like the scaling method, pupils often use it even when it has not been formally taught. It was not used by any pupils in the video, but this may well be because it involves calculating with fractions, rather than for any other reason. The unitary method involves working out a rate, which is the change in one quantity per unit of the other.

A potential stumbling block is at the first stage of the solution. In this example, it is necessary to see that it is the number of miles per kilometre that is required, not the number of kilometres per mile. Of course, in many problems the rate is given in the data, for example: 'How far will I cycle in $3\frac{1}{2}$ hours at an average speed of 12 mph?'

The algebraic method

For the algebraic method, there are several ways in which an equation can be formed. For example:

Let x = number of miles in 320 kilometres. Then

$$\frac{x}{320} = \frac{5}{8}$$

leading to $x = \frac{5}{8} \times 320$

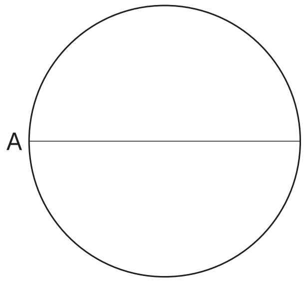
This is a more sophisticated method, appropriate to higher-attaining pupils, say in Year 9. It is necessary for more advanced work, for example in trigonometry.

Resource 8d Circle and diameter

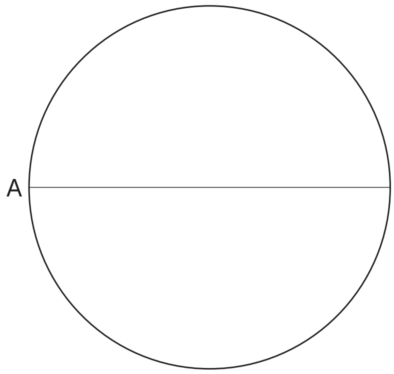
Look at the circle below.

Imagine a piece of string whose length is equal to the diameter, with one end fixed at A.

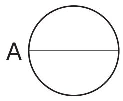
Imagine the string is wrapped in a clockwise direction around the circumference of the circle until it is taut. Mark the point that you think the other end of the piece of string will reach.



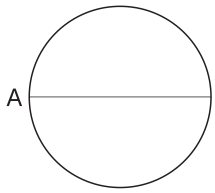
Repeat with this circle.



And this one.



And this one.



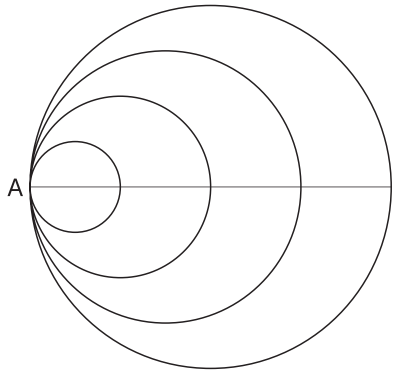
Do you agree that your marked point in each case is more than a quarter but less than half the circumference from A?

Let us say that the diameter is always roughly one third of the circumference, or:

— ≈ 3

This is an example of a proportion: the circumference of a circle is proportional to its diameter.

Now repeat the activity on the series of nested circles below.



What do you notice about your set of marked points on the set of nested circles?

What do you think pupils would gain from a task like this?

Resource 8e Metric paper sizes

You will need some sheets of A4 plain paper, scissors and glue, and a ruler.

1 Start with two A4 sheets of paper. By successive folding and cutting one of the sheets, produce smaller paper sizes – A5, A6, and (although not commercially recognised) A7, A8, A9, A10.

Put the seven paper sizes in order and align them, with their longer sides in the same direction, so that the sheets meet at a corner. Take this corner point as a centre of enlargement. Draw radiating lines from this corner as a visual check that the sheets represent successive enlargements.

Construct a table of measurements – length, width and length/width.

Confirm that the scale factor of enlargement is approximately 1.4.

2 A sheet of metric paper of size A0 has an area of 1 m^2. The next smaller size, A1, is obtained by halving an A0 sheet, by folding along a line parallel to the shorter side. Subsequent sizes, A2, A3, A4, A5 and A6, are produced in a similar way.

The dimensions of the A0 sheet are such that A1, A2, …, have their sides in the same proportion.

Find the ratio of length to width of a sheet of metric paper.

3 Calculate the length and width of a sheet of A4 paper, correct to the nearest millimetre.

Resource 8f Summary and further action on Module 8

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching of ratio, proportion and proportional reasoning.

List two or three key points that you have learned. • • •

List two or three points to follow up in further study. • • •

List two or three modifications that you will make to your planning or teaching of ratio, proportion and proportional reasoning. • • •

List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

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MODULE

9

Using and applying mathematics

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- considers the nature of problem solving in Key Stage 3 and the implications for planning and teaching;
- discusses examples on using and applying mathematics in the *Framework for teaching mathematics: Years 7, 8 and 9*;
- considers types of questions that will engage pupils in problem solving and probe their understanding.

CONTENT

The module is in six parts.

1 Introduction

2 Devising questions to probe pupils' understanding

3 Exploring the supplement of examples

4 Two problem-solving activities

5 Focusing teaching on objectives

6 Summary

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- The *Framework for teaching mathematics: Years 7, 8 and 9*
- A calculator
- A highlighter pen
- The resource sheets at the end of this module:

9a Examples of probing questions
9b Questioning pupils about percentages
9c Everything 15% off
9d Classifying quadrilaterals
9e Lesson 1: Year 8
9f Starter and plenary for Lesson 1
9g Lesson 2: Year 7
9h Starter and plenary for Lesson 2
9i Summary and further action on Module 9

Desirable

Assess and review lessons, which you can download from:
http://www.standards.dfes.gov.uk/primary/publications/mathematics/assess_review/

STUDY TIME

Allow approximately 2 to 2.5 hours

Part 1 Introduction

1 From the summer of 2003, Key Stage 3 mathematics test papers have included a greater emphasis on using and applying mathematics. The questions assess pupils' skills in communicating mathematically, reasoning and solving problems.

Module 9 looks at the teaching of problem solving and draws on examples for using and applying mathematics in the *Framework for teaching mathematics: Years 7, 8 and 9,* section 4.

Find page 20 of the Framework section 1. Read the first three paragraphs, including the four bullet points.

Consider how closely this description matches your approach to teaching pupils to:

- use and apply their mathematics in the context of problem solving;
- develop their general thinking skills.

In your personal file, comment on how you can ensure that your teaching incorporates the elements of a good 'diet' listed in the third paragraph.

Part 2 Devising questions to probe pupils' understanding

1 All teaching of mathematics is dependent on good questioning strategies. This is especially so in the teaching of problem solving.

The National Numeracy Strategy produced some guidance materials on probing questions. Read **Resource 9a, Examples of probing questions**, which relate to the key objectives for Year 6. As you study the questions, use a highlighter pen to indicate those that you could use with any of the Key Stage 3 classes that you teach.

---

2 Now try drafting some probing questions. Try the activity on **Resource 9b, Questioning pupils about percentages**.

As you develop and refine each question, think about these factors.

- How much will this question reveal about the depth of pupils' understanding and thinking?
- As pupils consider this question, will it move their thinking forward?
- Is this question sufficiently open to be asked of a large group or whole class, or would it be better addressed to an individual or small group of pupils?

---

3 Use the list of questions below to refine and add to your list of sample questions on Resource 9b.

- What percentages can you easily work out in your head? Talk me through a couple of examples.
- When calculating percentages of quantities, what percentage do you usually start from? How do you use this percentage to work out others?
- Are there any percentages that you cannot work out?
- 50% of the numbers on this 1–100 grid are even. How would you check?
- Give me a question with an answer of 20%.

- 'To calculate 10% of a quantity you divide it by 10. So to find 20% you must divide by 20.' What is wrong with this statement?

4 It is not easy to devise probing questions. Certain starting phrases can be helpful. For example:

- How do you know …? What do you look for?
  e.g. How do you know you have found the simplest form of a fraction? What do you look for?
- How do you go about …?
  e.g. How do you go about ordering a set of decimal numbers? What do you look for first? Which kinds of numbers are particularly difficult to order?
- Make up some questions …
  e.g. Make up some questions that have the answer 'two-fifths'
  … that have the answer 'a parallelogram'.
- Give me a —— that is the same as ——. How did you do it?
  e.g. Give me an equation that has the same solution as $6 = 2p - 8$. How did you do it?
- Is … always true, sometimes true, or never true? How do you know?
  e.g. Consider the statement $(3x)^2 = 3x^2$. Is this always true, sometimes true or never true? Explain your reasoning.
- What clues do you use when …?
  e.g. What clues do you use when you are trying to find the size of an unknown angle in a geometric diagram?

5 Carefully targeted probing questions are a vital assessment tool for any teacher. An important reason for using them regularly is that it helps pupils to see their power and to begin to formulate similar questions for themselves.

Pupils should also learn the habit of regularly asking themselves questions such as:

- What do I know that will help me here?
- Is that the best way of tackling this?
- How can I check that this is a reasonable answer?

These are high-level thinking skills that pupils need to develop as part of their problem-solving repertoire.

Part 3 Exploring the supplement of examples

1 We will now consider some examples of problems suitable for Key Stage 3 pupils.

In your *Framework for teaching mathematics: Years 7, 8 and 9,* turn to the Year 8 yearly teaching programme, section 3, page 8. Study the objectives for 'using and applying mathematics to solve problems'. Compare the Year 8 objectives with those for Year 7, page 6, and Year 9, page 10. Do you get a sense of the expected progression across Years 7, 8 and 9? Pay special attention to the key objectives, to which special emphasis should be given.

In your personal file, jot down all the verbs associated with using and applying mathematics (for example, *solve …*, *identify …*, *justify …*, *explain …*).

Think about how and when Key Stage 3 objectives for using and applying mathematics are taught in your school. Do they get enough emphasis?

If there are any points that you would like to discuss later with your head of department, allocate a page of your personal file for them and jot them down.

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2 The supplement of examples, Framework section 4, starts by illustrating 'using and applying mathematics to solve problems'. The first part of the section (pages 2–25) contains examples for solving word problems. These examples cover all strands of mathematics: number, algebra, shape, space and measures, and handling data. After the examples for word problems, other objectives for using and applying mathematics are exemplified on pages 26–35.

Turn to the supplement of examples, pages 16–17. Spend 10 minutes or so solving a small selection of the problems on these pages in your personal file.

If you have studied Modules 5 and 6, how do these problems and solutions link to the work you did on geometrical reasoning? Next to each of your solutions, make a note of the possible links with geometrical reasoning and how they could be made explicit to pupils.

---

3 Now turn to the supplement of examples, pages 26–27. The problems on these pages relate to the two objectives:

- Identify the information necessary to solve a problem.
- Represent problems mathematically in a variety of forms.

Choose one or two of the problems. Spend 10 minutes or so exploring the problems and considering possible solutions.

In your personal file, make a note of any of the problems that you have tried that you would like to give to pupils. Note also any implications for your planning and teaching, such as appropriate points of your school's scheme of work when the problems could be introduced, or the prerequisite skills that pupils would need.

---

4 Now go back and look again at each of the problems that you have considered above. For each problem, decide whether you would use it to follow up some previous teaching, so that pupils can use and apply their skills, or whether you would use it to introduce some new mathematics.

Once again, if there are any points that you would like to discuss later with your head of department, add them to the relevant page of your personal file. For example, you may be unsure how a problem could be used to launch a new mathematical topic – if so, your head of department will be able to advise you.

Part 4 Two problem-solving activities

1 There are two different types of problem solving:

- using and applying in a problem-solving context the mathematics that has already been taught;
- problems or starting points that enable pupils to explore and investigate new mathematics.

In thinking about these two types of problem solving, there are several points to note.

- Some problems require pupils to draw on objectives that they have already learned and to use them in a problem-solving context. These problems are often used in the middle or towards the end of a unit of work.
- Some problems are designed to motivate pupils to explore and develop understanding of new objectives. These problems can be used at any stage of a unit.
- Problems may be of short duration and completed in one lesson; others may require more sustained work and span two or more lessons.

You will now work on two activities taken from the Key Stage 3 Framework, section 4, the supplement of examples. The activities have been chosen to illustrate the two main types of problem solving and to link back to Modules 5 to 8, which focused on proportional and geometrical reasoning.

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2 First, explore the problem on **Resource 9c, Everything 15% off**.

Then try the problem on **Resource 9d, Classifying quadrilaterals**.

---

3 Possible lines of questioning that you could use with these two problems could include:

- **Everything 15% off**

  What could you change in the original problem in order to explore further?
  For example:

  – exploring for any percentage reduction;

  – exploring for any percentage increase;

  – exploring what happens when an amount is increased by a percentage and then increased again, or vice versa.

  Which is better: a 10% reduction before VAT is added, or a 10% reduction after VAT is added? Justify your answer.

- **Classifying quadrilaterals**

  Why is it not possible to put a quadrilateral in this cell?

  What other pair of categories could you use to classify quadrilaterals?

  What about classifying other shapes?

Part 5 Focusing teaching on objectives

1 Effective teaching is based on clear objectives. Look again at the objectives for the two activities that you have been working on, shown on Resources 9c and 9d. You are now going to plan two lessons, each incorporating one of these activities. This will involve thinking about and making notes on *how to teach* the objectives.

Before you begin to plan the lessons, read ‘The focus on direct teaching’, in the Framework section 1, pages 26–27.

---

2 Direct teaching does not necessarily have to happen before pupils start to work on the activity. You may feel it is appropriate for pupils to begin to explore the activity before teaching the whole class.

While you plan the direct teaching for the main part of your lessons, you may need to:

- focus on particular elements of direct teaching, such as demonstrating and explaining, questioning and discussing;
- think about how you will ensure that all pupils gain from the teaching and are able to tackle the problem;
- think about the specific questions that you will use;
- think about the kinds of resources that would be appropriate.

Start by planning the direct teaching for the main part of the lesson. Begin with the Year 8 lesson, and the problem of 15% off everything. Complete the relevant sections on **Resource 9e, Lesson 1: Year 8**, or, if you prefer, design your own format in your personal file.

---

3 Now plan an oral and mental starter and plenary for your lesson. Use **Resource 9f, Starter and plenary for Lesson 1**, or design your own format in your personal file. Refer back to your notes on probing questions as you do this, particularly those on percentages.

---

4 Next, repeat what you have done, this time focusing on the classifying quadrilaterals problem for Year 7. Use **Resource 9g, Lesson 2: Year 7**, and **Resource 9h, Starter and plenary for Lesson 2**, or design you own format in your personal file.

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5 If possible, discuss your plans with your head of department, a Key Stage 3 consultant, an advisory teacher or an experienced colleague. Ask them for feedback on:

- whether the plans would be effective in teaching pupils to use and apply mathematics (with specific reference to the particular objectives of the lesson);
- whether the plans cater sufficiently well for the different needs and abilities of the pupils in the class that you have in mind;
- whether the questions that you will use will support pupils, probe their thinking and help to extend their problem-solving skills.

Part 6 Summary

1 You can introduce problem solving and opportunities to use and apply mathematics at many points in a unit of work. A problem can serve as an introduction, to assess pupils' prior knowledge or to set a context for the work; it can be used to provide motivation for acquiring a skill; or it can be set as a class activity or as homework towards the end of a topic, so that pupils use and apply the mathematics they have been taught.

Aim to provide a range of opportunities that include:

- problems and applications that extend content beyond what has just been taught;
- familiar and unfamiliar problems in a range of numerical, algebraic and graphical contexts, some with a single solution and some with several possible solutions;
- activities that develop short chains of deductive reasoning and concepts of proof in algebra and geometry;
- occasional opportunities to sustain thinking by tackling more complex problems.

Wherever possible, use these opportunities to help pupils to appreciate the connections between different aspects of mathematics.

2 A teacher's questions are central to the development of pupils' reasoning. Good questions prompt pupils to analyse, justify and evaluate their problem-solving strategies.

Several different prompts can be useful in probing pupils' thinking.

- How do you go about …?
- What do you look for when …?
- How do you know that …?
- Why do you think that …?
- How did you reach that conclusion?
- Can you explain why that is right?
- What might explain that?
- How is that possible?
- Is there another way?
- What explanation do you think is best?
- Does it always work? Why?
- Is it always true, sometimes true, or never true? Why?

3 Look back over the notes you have made during this module. Have you identified all the factors that you want to consider and adopt when you are planning and teaching pupils to use and apply their mathematics and to solve problems?

Use **Resource 9i, Summary and further action on Module 9**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and any points to discuss with your head of department.

4 If you are interested in looking at more examples of the National Numeracy Strategy's suggestions for probing questions, download *Assess and review lessons* from http://www.standards.dfes.gov.uk/primary/publications/mathematics/assess_review/.

You might also wish to read the leaflet *Changes to assessment 2003: guidance for teachers of KS3 mathematics* (ref: QCA/02/980).

Resource 9a Examples of probing questions

<table>
<tr><th>Objective</th><th>Sample probing questions</th></tr>
<tr><td>Multiply and divide decimals mentally by 10 or 100, and integers by 1000, and explain the effect.</td><td>

Why do 5 ÷ 10 and 50 ÷ 100 give the same answer?
This calculator display shows 0.1. Tell me what will happen when I multiply by 100. What will the display show?
What number is ten times as big as 0.01? How do you know that it is ten times 0.01?
I divide a number by 10, and then again by 10. The answer is 0.3. What number did I start with? How do you know?
How would you explain to someone how to multiply a decimal by 10?
</td></tr>
<tr><td>Order a mixed set of numbers with up to three decimal places.</td><td>

What did you look for first?
Which part of each number did you look at to help you?
Which numbers did you think were the hardest to put in order? Why?
What do you do when numbers have the same digit in the same place?
Can you explain this to me using a number line?
Give me a number somewhere between 3.12 and 3.17. Which of the two numbers is it closer to? How do you know?
</td></tr>
<tr><td>Reduce a fraction to its simplest form by cancelling common factors.</td><td>

What clues did you look for to cancel these fractions to their simplest form?
How do you know when you have the simplest form of a fraction?
Give me a fraction that is equivalent to $\frac{2}{3}$, but has a denominator of 18. How did you do it?
</td></tr>
<tr><td>Use a fraction as an operator to find fractions of numbers or quantities (e.g. $\frac{5}{8}$ of 32, $\frac{7}{10}$ of 40, $\frac{9}{100}$ of 400 cm).</td><td>

$\frac{2}{5}$ of a total is 32. What other fractions of the total can you calculate?
Using a set of fraction cards (e.g. $\frac{3}{5}$, $\frac{7}{8}$, $\frac{5}{6}$, $\frac{3}{4}$, $\frac{7}{10}$, etc.) and a set of two-digit number cards, ask how the fractions and numbers might be paired to form a question with a whole-number answer. Ask: What clues did you use?
</td></tr>
<tr><td>Solve simple problems involving ratio and proportion.</td><td>

There are 20 boys and 10 girls in Class 6. Give me a sentence using the word ‘ratio’ (or ‘proportion’). What other possibilities are there?
Look at this Carroll diagram.
<table>
<tr><th></th><th>Boys</th><th>Girls</th></tr>
<tr><td>Wear glasses</td><td>1</td><td>3</td></tr>
<tr><td>Don’t wear glasses</td><td>2</td><td>4</td></tr>
</table>
Give me a question that has the answer:
3 : 7
40%
2 : 1
$\frac{2}{3}$
</td></tr>
</table>

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Objective	Sample probing questions
Carry out column addition and subtraction of numbers involving decimals.	• Make up an example of an addition/subtraction involving decimals that you would do in your head and one you would do on paper. Explain why. • Give pupils some completed questions to mark. Questions need to be written horizontally as well as in column form. Include incorrect answers such as 12.3 + 9.8 = 21.11; 4.07 – 1.5 = 3.92; 3.2 – 1.18 = 2.18. Ask: Which are correct/incorrect? How do you know? Explain what has been done wrong and correct the answers.
Derive quickly division facts corresponding to multiplication tables up to 10×10.	• Start from a two-digit number with at least 6 factors, e.g. 56. How many different multiplication and division facts can you make using what you know about 56? How have you identified the different divisions? What if you started with 5.6? What about 11.2? Or 1120?
Carry out 'short' multiplication and division of numbers involving decimals.	• The answer is 12.6. Make up some questions using multiplication and division with decimal numbers.
Carry out long multiplication of a three-digit by a two-digit integer.	• Give pupils three or four long multiplications with mistakes in them. Ask them to identify the mistakes and talk through what is wrong and how they should be corrected. • Give pupils a multiplication question (for example, 147×32) calculated by both the grid method and long multiplication. Ask questions such as: What two numbers multiplied together give 4410? Or 294?
Use a protractor to measure acute and obtuse angles to the nearest degree.	• Ask pupils to estimate and measure a range of acute and obtuse angles using a transparency of a protractor with the numbers removed. • As above, but with the two corners broken off. • What important tips would you give to a person about using a protractor?
Calculate the perimeter and area of simple compound shapes that can be split into rectangles.	• Why is it a good idea to split this shape into rectangles to find the area? • How do you go about calculating the dimensions of the rectangles? … the compound shape? • Form a compound shape by pushing together two rectangles. Compare the area and perimeter of the rectangles with the compound shape. What has changed and why? What happens if you join the rectangles in a different way? Why?
Read and plot coordinates in all four quadrants.	• A square has vertices at (0, 0), (3, 0) and (3, 3). What are the coordinates of the fourth vertex? • A square has vertices at (3, 0), (0, 3) and ($^-3$, 0). What are the coordinates of the fourth vertex? • A square has vertices at (0, 0) and (2, 2). Give two possible answers for the positions of the other two vertices. • A square has vertices at ($^-1$, 1) and ($^-2$, $^-3$). Give two possible answers for the positions of the other two vertices.

Objective	Sample probing questions
Identify and use the appropriate operations (including combinations of operations) to solve word problems involving numbers and quantities, and explain methods and reasoning.	• What clues do you look for in the wording of questions? What words mean you need to add, subtract, multiply or divide? • Make up two different word problems for each of these calculations. Try to use a variety of words. $(17 + 5) \times 6$ $12.5 \div 5 - 0.25$
Solve a problem by extracting and interpreting information presented in tables, graphs and charts.	• From a given graph/table/chart, make up three questions that can be answered using the graph/table/chart. • What makes the information easy or difficult to interpret?

Resource 9b Questioning pupils about percentages

Imagine that you are planning a lesson on percentages for a mixed-ability Year 7 class. The objective of the lesson is:

- **Understand percentage as 'the number of parts per 100'; recognise the equivalence of percentages, fractions and decimals; calculate simple percentages and use percentages to compare simple proportions.**

In the space below, make a note of half a dozen questions that you could use in the final plenary of such a lesson, working with the whole class.

The questions should probe pupils' thinking and help you to assess how well they have absorbed the lesson.

Sample probing questions

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Resource 9c Everything 15% off

Problem (see the Framework section 4, page 29, Year 8)

In a gift shop sale everything is reduced by 15%.
A quick way of calculating the sale price is to multiply the original price by a number.
What is the number?
Give a mathematical reason to justify your answer.

After two weeks, the sale price is reduced by a further 15%.
Show that this means the original price has been reduced by 27.75%.

Main objective

- Solve more complex problems by breaking them into smaller steps, choosing efficient techniques for calculation, algebraic manipulation and graphical representation, and choosing suitable resources, including ICT.

Related objectives

- Use logical argument to establish the truth of a statement; give solutions to an appropriate degree of accuracy in the context of the problem.

Links to other strands/objectives

- Express a given number as a percentage of another; use the equivalence of fractions, decimals and percentages to compare proportions; calculate percentages and find the outcome of a given percentage increase or decrease (Framework section 4, pages 70–77).

Possible approaches or methods to use for this problem and potential for exploring it further:

[continued on the next page]

Possible questions that you could use to support, extend or challenge pupils' thinking:

Resources or special preparation that would be needed:

Other notes:

Resource 9d Classifying quadrilaterals

Problem (see the Framework, section 4, page 34, Year 7)

Copy the table below onto a large piece of paper.
Draw and name quadrilaterals in the appropriate spaces.

		Number of pairs of parallel sides		
		0	1	2
Number of pairs of equal sides	0			
	1			
	2			

Will any of the spaces remain empty? If so, explain why.

Main objective

- Suggest extensions to problems by asking 'What if …?'; begin to generalise and to understand the significance of a counter-example.

Related objectives

- Identify the necessary information.
- Represent problems mathematically, making correct use of symbols, words, diagrams, tables or graphs.

Links to other strands/objectives

- Identify and use angle, side and symmetry properties of triangles and quadrilaterals (Framework section 4, pages 184–189).

Possible approaches or methods to use for this problem:

[continued on the next page]

Possible questions that you could use to support, extend or challenge pupils' thinking:

Resources or special preparation that would be needed:

Other notes:

Resource 9e Lesson 1: Year 8

Main activity

Objectives
• Solve more complex problems by breaking them into smaller steps, choosing efficient techniques for calculation, algebraic manipulation and graphical representation, and choosing suitable resources, including ICT. • Use logical argument to establish the truth of a statement; give solutions to an appropriate degree of accuracy in the context of the problem. • Express a given number as a percentage of another; use the equivalence of fractions, decimals and percentages to compare proportions; calculate percentages and find the outcome of a given percentage increase or decrease.

Key vocabulary

Activity
Problem In a gift shop sale everything is reduced by 15%. A quick way of calculating the sale price is to multiply the original price by a number. What is the number? Give a mathematical reason to justify your answer. After two weeks, the sale price is reduced by a further 15%. Show that this means the original price has been reduced by 27.75%.

Notes on direct teaching for main part of lesson
[continued on the next page]

Notes on direct teaching for main part of lesson (continued)

Resource 9f Starter and plenary for Lesson 1

Plan an oral and mental starter and a plenary for Lesson 1.

Starter for Lesson 1

Teaching group and objective(s)

Key vocabulary

Notes on organisation, activity, key questions

[continued on the next page]

Plenary for Lesson 1

Plenary organisation and activity

Examples of probing questions to use in the plenary

Key points for pupils to remember to be drawn out at end of plenary

Main activity

Objectives

- Suggest extensions to problems by asking 'What if …?'; begin to generalise and to understand the significance of a counter-example.
- Identify the necessary information.
- Represent problems mathematically, making correct use of symbols, words, diagrams, tables or graphs.
- Identify and use angle, side and symmetry properties of triangles and quadrilaterals.

Key vocabulary

Activity

Problem

Copy the table below onto a large piece of paper.
Draw and name quadrilaterals in the appropriate spaces.

Will any of the spaces remain empty? If so, explain why.

		Number of pairs of parallel sides		
		0	1	2
Number of pairs of equal sides	0			
	1			
	2			

Notes on direct teaching for main part of lesson

[continued on the next page]

Notes on direct teaching for main part of lesson (continued)

Resource 9h Starter and plenary for Lesson 2

Plan an oral and mental starter and a plenary for Lesson 2.

Starter for Lesson 2

Teaching group and objective(s)

Key vocabulary

Notes on organisation, activity, key questions

[continued on the next page]

Plenary for Lesson 2

Plenary organisation and activity

Examples of probing questions to use in the plenary

Key points for pupils to remember to be drawn out at end of plenary

Resource 9i Summary and further action on Module 9

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching of using and applying mathematics.

List two or three key points that you have learned.

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List two or three points to follow up in further study.

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List two or three modifications that you will make to your planning or teaching of using and applying mathematics.

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List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

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MODULE

10

Effective oral and mental work

OBJECTIVES

This module is for study by an individual teacher or group of teachers. It:

- considers the importance of oral and mental work in all parts of mathematics lessons;
- discusses how to develop a programme of oral and mental starters.

CONTENT

The module is in four parts.

1 Introduction

2 Oral and mental starters

3 Oral and mental work in the main part of the lesson and the plenary

4 Summary

RESOURCES

Essential

- Your personal file for inserting resource sheets and making notes as you work through the activities in this module
- The *Framework for teaching mathematics: Years 7, 8 and 9*
- Video sequence 6, Oral and mental starters, from the CD-ROM accompanying this module
- The resource sheets at the end of this module:

  10a Features of oral and mental starters

  10b Reflections

  10c Plan for an oral and mental starter

  10d Equilateral triangles

  10e Types of question

  10f How children travel to school

  10g Transformations

  10h Plan for a plenary

  10i Summary and further action on Module 10

Desirable

The National Numeracy Strategy publication *Mathematical vocabulary*, which you can download from:
http://www.standards.dfes.gov.uk/primary/publications/mathematics/vocabulary/

STUDY TIME

Allow approximately 90 minutes.

Part 1 Introduction

1 This module emphasises the importance of oral and mental work in all parts of a mathematics lesson. Effective oral and mental work is important because:

- it is the basis of good interactive teaching;
- it engages and motivates pupils;
- it models and practises the speaking, listening, discussion and thinking skills which pupils need to develop.

The first half of the module discusses how to establish a regular programme of oral and mental starters to lessons. In the second half, we look at the use of oral and mental work in the main part of the lesson and the plenary.

Part 2 Oral and mental starters

1 There are several purposes of a short, focused oral and mental activity at the start of a lesson. The activity may be designed to fulfil one or more of these aims:

- To ensure that the lesson gets off to a purposeful start and sets a brisk pace
  For example, having a puzzle on the board as pupils arrive can be a good settling activity (e.g. arithmagons, magic squares with algebra, graphs that need explaining). Once pupils have had a chance to work on the puzzle, discuss the strategies they have used.
- To rehearse previously taught skills in a variety of lively ways
  The wide range of skills that can be practised include mental calculations of all types, simple algebraic manipulations, estimations of measurements and calculations, visualisation skills involving imagery, interpretation of graphs and charts, and so on. It is all too easy to forget facts if you don't have regular opportunities to recall them. Occasional practice of recall of prime numbers to 100, addition and subtraction facts to 20, conversions of units of measurement (including time and speed), fraction, decimal and percentage equivalents, multiplication and division facts and so on fall into this category.
- To focus on the skills needed in the main part of the lesson
  If you pre-empt problems and rehearse skills at the start of the lesson, pupils are able to work without interruption later on. Examples might be practising cancelling fractions as preparation for work on probability; doing calculations such as $12 \div 0.5$ before calculating coordinates of the graph $y = 12/x$.
- To make use of homework from a previous lesson in an introductory activity
  For example, pupils may have collected some data and may discuss at the start of a lesson how best to organise it; they may have done some calculations in preparation for comparisons of calculation methods, or solved a geometric problem in readiness for sharing explanations of their reasoning.
- To make an informal assessment of pupils' understanding and progress in order to inform the direction of the next part of the lesson
  Making sure that you get feedback from a high proportion of pupils – for example, by using mini-whiteboards or by using targeted questions – allows you to make a rapid informal assessment of the group. This can help ensure that the lesson is pitched at an appropriate level so that pupils' knowledge and understanding can be consolidated and extended. It also provides an opportunity for misconceptions to be identified and rectified immediately or noted for tackling later.

2 You now have an opportunity to watch a selection of oral and mental starters. Get ready to watch **Video sequence 6, Oral and mental starters**. While you are watching, note any examples of the features listed on **Resource 10a, Features of oral and mental starters**.

Watch the video sequence, which lasts about 8 minutes.

When you have finished watching, spend a few minutes completing the notes you have made on Resource 10a.

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3 Item 1 above described a number of general purposes of oral and mental starters. The starters can also be used to help meet a wide range of mathematical objectives. For example:

- Develop and explain mental calculation strategies, including figuring out new facts from known facts and explaining the strategies used
  A strong emphasis on explaining strategies helps other pupils to increase their repertoire of skills. Pupils should be encouraged to consider the efficiency, reliability and applicability of their own and others' strategies.
- Apply number facts to real-life situations
  Pupils should apply number facts and calculation strategies to solve word problems and real-life problems which involve more than one step, money problems and problems using units of measurement.
- Develop estimation skills
  Developing estimation skills is an important element in the use of calculators. Pupils need quick recall of number facts when dealing with decimals – for example, estimating 0.221×5.17 as $0.2 \times 5 = 1$ to check that the calculator answer is of the correct magnitude.
- Develop links between the laws of number and those of algebra
  The Key Stage 3 Framework has an important message: that algebra is generalised arithmetic. Pupils should have a good grasp of solving algebraic equations mentally before progressing to more formal methods.
- Develop mental imagery of shapes, movements and constructions
  Pupils need to be able to visualise geometrical shapes, consider their properties and relationships, and analyse their transformations, giving reasons for their results and conclusions. These skills underpin the development of geometrical proof.
- Develop inference skills from data in a variety of forms
  Pupils need to be able to analyse and make inferences from data. The oral and mental starter is a good time to ask pupils to look at graphs, charts and tables, to describe their observations, and to justify their conclusions or hypotheses.
- Develop the use of correct mathematical vocabulary
  Pupils need to use correct mathematical vocabulary and notation, and understand the meaning of mathematical terms. They may fail to answer correctly because they are not familiar with the correct terminology. This can give rise to confusion between, for example, an expression and an equation, and the instruction to simplify or to solve.
- Develop the ability to generalise, reason and prove
  Pupils need to develop the ability to generalise, reason and prove. They should be able to give a justification for an answer and to see the difference between demonstration and proof.

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4 Use **Resource 10b, Reflections**, to consider the features identified in paragraph 3 above. Which of them do you use regularly in your own lesson starters and which might you think about using more frequently?

5 Section 4, the supplement of examples, of the Key Stage 3 Framework has a wide range of examples of activities that can be used as oral and mental starters.

Use **Resource 10c, Plan for an oral and mental starter**, to design an oral and mental starter activity of a type that you don't use regularly. You will first need to identify a learning objective for the activity and a class with whom to trial it.

Part 3 Oral and mental work in the main part of the lesson and the plenary

1 Many schools have gone a long way towards establishing regular oral and mental starters to lessons. It is equally important to engage pupils in effective oral and mental work in later parts of the lesson.

Many lessons, though not all, will involve some focused teaching at the start of the main part. Where the lesson is partway through a topic, this might be in the form of a short reminder about what happened previously, stressing key ideas, vocabulary and particular techniques, for example. It will also be the time to ensure that all pupils know what they are to work on in this lesson. At the start of new topics and at key points throughout them, the direct teaching input will be more substantial. In each of these situations, pupils should be deploying and practising their oral and mental skills through interactions with their teacher.

We are going to look at three examples of main teaching activities that focus on particular oral and mental skills.

2 **Example 1: Solving problems in geometry**

In the first example, Year 8 pupils practise their listening skills and powers of mental imagery. The objectives for main part of the lesson are that pupils will be taught to:

- solve geometrical problems using side and angle properties of equilateral triangles, explaining reasoning with diagrams and text;
- use logical argument to establish the truth of a statement.

Read through the 'script' in **Resource 10d, Equilateral triangles**. Try to put yourself in the shoes of a pupil, and respond accordingly.

3 In a real class, the introduction to the main activity on Resource 10d ought to engage everyone in thinking about the properties of shapes. It should also give you the chance to make a quick assessment of pupils' confidence and familiarity with the ideas and the language. At the conclusion of this activity, the class should be aware that three different shapes can be made from four equilateral triangles. Two of these three shapes form the net of a regular tetrahedron.

Now consider what the next part of the lesson might look like. Make some notes on how it might develop in your personal file.

4 One possibility for further development would be to address the question: 'How do we know when we have found all the possible shapes made from four equilateral triangles?'. Another would be to extend the idea to five triangles and ask pairs or groups of pupils to classify their findings according to a criterion they choose (for example, the order of rotation symmetry).

5 **Example 2: Data handling**

The second example focuses on the design and use of questions to engage pupils' thinking.

Before you look at the example itself, read **Resource 10e, Types of question**. This is based on the National Numeracy Strategy publication *Mathematical vocabulary*, which you can download from the DfES Standards website (http://www.standards.dfes.gov.uk/primary/publications/mathematics/vocabulary/).

6 We will now consider a Year 7 class that has been collecting data and representing it using a computer spreadsheet. The teacher wants to begin exploring appropriate and inappropriate use of computer charts. The unit's objectives include: 'interpret diagrams and graphs, and draw inferences'.

Look at **Resource 10f, How children travel to school**. The two graphs are different ways of representing the same set of data. Imagine that you have shown the graphs to a Year 7 class. Use Resource 10f to jot down some questions that will establish how well pupils are able to interpret the first graph. Then jot down some questions that would encourage pupils to compare the relative usefulness of the two graphs.

7 Do your suggested questions related to the bar chart include both open and closed questions?

Examples of closed questions might include:

- *How many boys travelled by bus?*
- *How many more girls than boys travelled by car?*
- *How many children were in the survey?*
- *About a quarter of pupils used one form of transport. Which was it?*

Examples of open questions might include:

- *What does the graph show?*
- *Approximately what percentage of pupils walked? How did you work it out?*
- *How can you be sure?*

Questions which would encourage pupils to compare the relative usefulness of the two graphs might include:

- *Which is the better way to present this information and why?*
- *On the line graph why are the points joined by a line? Is there a meaning to values between the points?*
- *How would you decide which graph to use?*

8 **Example 3: Transformation geometry**

In this third example, we will look at how to generate meaningful group discussion. It is taken from a Year 9 class who are coming to the end of a unit on transformation geometry. This included the objective: 'recognise and visualise transformations and symmetries of 2-D shapes'.

Resource 10g, Transformations, shows a diagram that is provided as part of the department's resources for the unit. Assume that you want to use it to assess both how well the pupils understand the different transformations they have met and how well they can use the associated mathematical terms. Use Resource 10g to jot down some probing questions designed to encourage discussion among pupils in groups. The questions should help you to listen in and assess pupils' understanding and use of language.

9 During longer lessons, it can be useful to bring pupils together to gauge how they are progressing and to check for misconceptions. Often, though not always, such plenaries will be at the end of lessons. A suitable oral and mental session at this point enables a teacher to do a number of things: assess progress; identify misconceptions; decide whether pupils are repeating an error; and consider whether they are ready to move on or require further teacher input to make progress. All this helps to ensure that a good pace is maintained and that teaching remains appropriately targeted.

Use **Resource 10h, Plan for a plenary**. Using Example 3 above, consider how you would bring the class together for a plenary designed to confirm what pupils have learned and to reveal any remaining misconceptions.

Part 4 Summary

1 Improving the quality of pupils' oral and mental skills is a key objective in improving standards of mathematics learning nationally. Many departments have identified improving questioning and discussion as targets for development work.

If possible, discuss your plans for an oral and mental starter and plenary with your head of department, an advisory teacher or an experienced colleague. Ask for feedback on:

- whether the plans would be effective in teaching pupils to articulate their ideas and explain and justify their reasoning;
- whether the plans cater sufficiently well for pupils' different needs and abilities;
- whether the questions that you will use will probe pupils' thinking and help to extend their oral and mental skills.

2 Look back over the notes you have made during this module. Have you identified all the factors that you want to consider and adopt when you are thinking about oral and mental work?

Use **Resource 10i, Summary and further action on Module 10**, to list key points you have learned, points to follow up in further study, modifications you will make to your planning or teaching, and any points to discuss with your head of department.

3 To follow up your work on these modules, you could try to visit a lesson taught by a leading teacher. Ask your head of department about your LEA's arrangements for this.

Resource 10a Features of oral and mental starters

Note examples from **Video sequence 6, Oral and mental starters**.

To what extent did teachers in the video sequence:

- have high expectations of pupils?
- provide a purposeful start to the lesson?
- identify clear learning outcomes?
- ensure that all pupils can and do take an active part in the session?
- prepare a good range of suitably differentiated open and closed questions to ask the class?
- target individuals, pairs or small groups with particular questions?
- use pupils’ responses to assess understanding and progress?

Resource 10b Reflections

I use oral and mental starters:	Often	Sometimes	Rarely	I'd like to strengthen this feature of my teaching
to develop and explain mental calculation strategies, including figuring out new facts from known facts and explaining the strategies used				
to apply number facts to real-life situations				
to develop estimation skills				
to develop links between the laws of arithmetic and those of algebra				
to develop mental imagery of shapes, movements and constructions				
to develop inference skills from data in a variety of forms				
to develop the use of correct mathematical vocabulary				
to develop the ability to generalise, reason and prove				

In this space, note any actions that you need to take to strengthen the features you have selected. Note also any points that you would like to discuss with your head of department.

Resource 10c Plan for an oral and mental starter

Teaching group and objective(s)

Key vocabulary

Notes on organisation, activity, key questions

Resource 10d Equilateral triangles

Read through the instructions below, a teacher's 'script' for the first part of the main teaching activity in a Year 8 geometry lesson. Try to put yourself in the shoes of a pupil, and respond accordingly. Don't rush – give yourself plenty of time to think. You should avoid any drawing or sketching and use mental images only.

The right-hand column is for you to note your responses to the bulleted questions, **using words only**. (In a real lesson, the teacher would ask one or more pupils to respond orally, using correct terminology.)

<table>
<tr><td>Imagine an equilateral triangle.
Imagine another identical equilateral triangle. Place it alongside the original so that the edges match exactly.
• What is the name of the shape you have made?
• Is there more than one possibility?
• Explain how you know.</td><td></td></tr>
<tr><td>Take another identical equilateral triangle and add this to the figure.
• What is the name of the shape you have made?
• Describe its properties as accurately as you can.
• Is there more than one possibility?
• Explain how you know.</td><td></td></tr>
<tr><td>Take four identical equilateral triangles.
How many different shapes can you make by joining edges?</td><td></td></tr>
<tr><td>Describe as accurately as possible one of the shapes you have made from the four equilateral triangles.
(In a real lesson, other pupils would be expected to sketch on their mini-whiteboards the shape being described. The shapes would then be revealed and compared.)</td><td></td></tr>
<tr><td>• What is the name of the shape you have made?
• Describe its properties as accurately as you can.
• What are the symmetries of your shape?
• Could it form the net of a regular polyhedron?</td><td></td></tr>
</table>

Consider whether this script would work if equilateral triangles were replaced by squares or by right-angled triangles.

Resource 10e Types of question

The use of questions is crucial in helping pupils to understand mathematical ideas and use mathematical terms correctly. Different sorts of question stimulate different sorts of thinking in pupils. It is easier to use certain types of question – those that ask the listener to recall and apply facts – than those that require a higher level of thinking. It is important to plan to use a full range of questions in your lessons.

Recalling facts

What is 3 add 7?

How many days are there in a week?

How many centimetres are there in a metre?

Is 31 a prime number?

Applying facts

Tell me two numbers that have a difference of 12.

What unit would you choose to measure the width of the table?

What are the factors of 42?

Hypothesising or predicting

Estimate the number of marbles in this jar.

If we did our survey again on Friday, how likely is it that our graph would be the same?

Roughly, what is 51 times 47?

How many crosses in the next diagram? + ++ +++
And the next?

Designing and comparing procedures

How might we count this pile of sticks?

How could you subtract 37 from 82?

How could we test a number to see if it is divisible by 6?

How could we find the 20th triangular number?

Are there other ways of doing it?

Interpreting results

So what does that tell us about numbers that end in 5 or 0?

What does the graph tell us about the most common shoe size?

So what can we say about the sum of the angles in a triangle?

Applying reasoning

The seven coins in my purse total 23p. What could they be?

In how many different ways can four children sit at a round table?

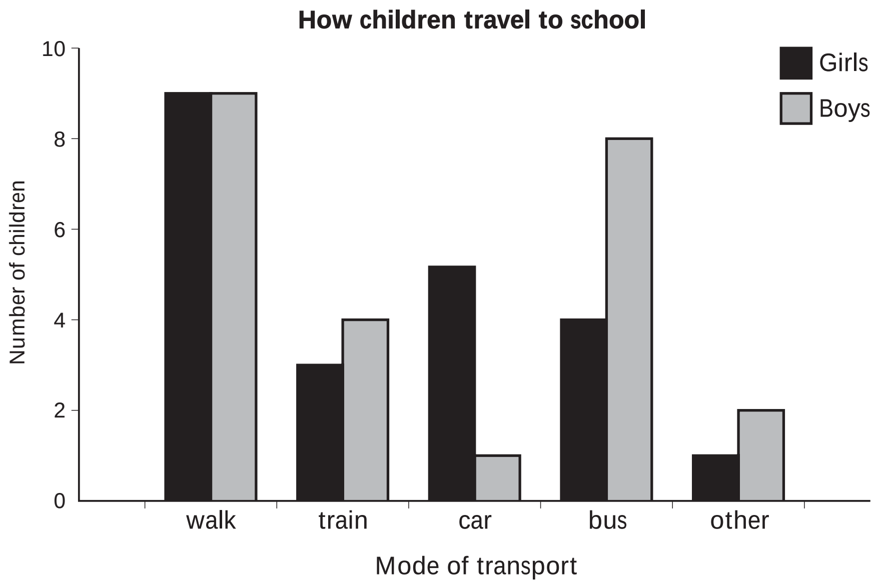
Why is the sum of two odd numbers always even?

This classification and the examples are taken from the National Numeracy Strategy guide *Mathematical vocabulary*, which you can download from:

http://www.standards.dfes.gov.uk/primary/publications/mathematics/vocabulary/

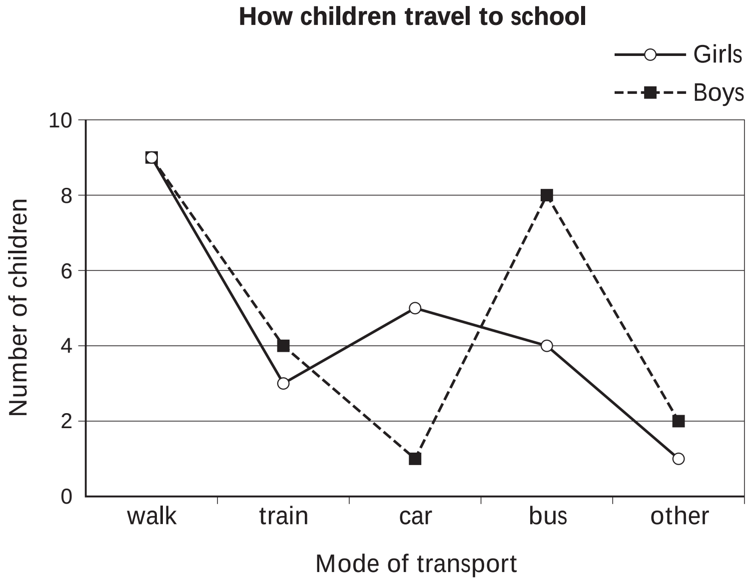
Resource 10f How children travel to school

The chart below represents how a group of children travel to school. Imagine that you have shown this graph to a Year 7 class.



Use this space to suggest questions that will establish how well pupils can interpret this type of graph.

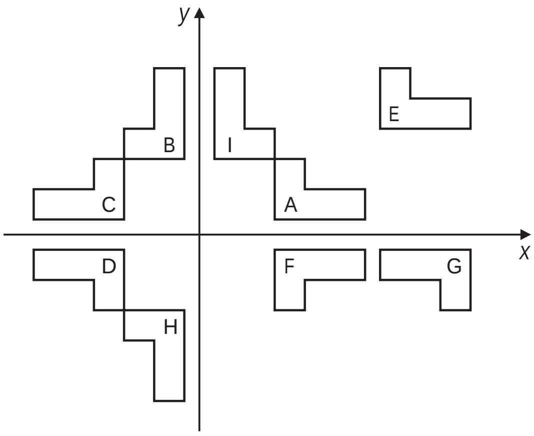
This is another way of representing the same data.



Use this space to suggest questions that would encourage pupils to compare the relative usefulness of the two graphs.

Resource 10g Transformations

Assume that the diagram below is provided as part of your department's resources for a Year 9 unit of work on transformations. The unit includes the objective: 'recognise and visualise transformations and symmetries of 2-D shapes'.



Assume that pupils are coming towards the end of the unit. You want to use the diagram to assess both how well the pupils understand the different transformations they have met and how well they can use the associated mathematical terms.

Use the space below to jot down some probing questions designed to encourage discussion among pupils in groups. The questions should help you to listen in and assess pupils' understanding and use of language.

Resource 10h Plan for a plenary

Plan a plenary designed to confirm what Year 9 pupils have learned about transformations and to reveal any remaining misconceptions.

Plenary organisation and activity

Examples of probing questions to use in the plenary

Key points for pupils to remember to be drawn out at end of plenary

Resource 10i Summary and further action on Module 10

Look back over the notes you have made during this module. Identify the most important things to consider and modify in your planning and teaching of oral and mental work.

List two or three key points that you have learned.

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List two or three points to follow up in further study.

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List two or three modifications that you will make to your planning or teaching of oral and mental work.

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List the most important points that you want to discuss with your head of department, or any further actions you will take as a result of completing this module.

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