DRAFT

Department for Education

Further mathematics

GCE AS and A level subject content

July 2014

Contents

The content for further mathematics AS and A levels

Introduction

1. AS and A level subject content sets out the knowledge, understanding and skills common to all specifications in further mathematics.

Purpose

2. Further mathematics is designed for students with an enthusiasm for mathematics, many of whom will go on to degrees in mathematics, engineering and the physical sciences.

3. The qualification is both deeper and broader than A level mathematics. It is normally taken as an extra subject, typically as a fourth A level. A level further mathematics builds from GCSE level and A level mathematics. As well as building on algebra and calculus introduced in A level mathematics, the A level further mathematics core content introduces complex numbers and matrices, fundamental mathematical ideas with wide applications in mathematics, engineering, physical sciences and computing. The non-core content includes different options that can enable students to specialise in areas of mathematics that are particularly relevant to their interests and future aspirations. A level further mathematics prepares students for further study and employment in highly mathematical disciplines that require knowledge and understanding of sophisticated mathematical ideas and techniques.

4. AS further mathematics, which can be co-taught with the A level as a separate qualification, is a very useful qualification in its own right, broadening and reinforcing the content of A level mathematics, introducing complex numbers and matrices, and giving students the opportunity to extend their knowledge in applied mathematics. This is very valuable for supporting the transition to degree level work and employment in mathematical disciplines.

Aims and objectives

5. AS and A level specifications in further mathematics must encourage students to:

- understand mathematics and mathematical processes in a way that promotes confidence, fosters enjoyment and provides a strong foundation for progress to further study
- extend their range of mathematical skills and techniques

- understand coherence and progression in mathematics and how different areas of mathematics are connected
- apply mathematics in other fields of study and be aware of the relevance of mathematics to the world of work and to situations in society in general
- use their mathematical knowledge to make logical and reasoned decisions in solving problems both within pure mathematics and in a variety of contexts, and communicate the mathematical rationale for these decisions clearly
- take increasing responsibility for their own learning and the evaluation of their own mathematical development
- reason logically and recognise incorrect reasoning
- generalise mathematically
- construct mathematical proofs
- use their mathematical skills and techniques to solve challenging problems which require them to decide on the solution strategy
- recognise when mathematics can be used to analyse and solve a problem in context
- represent situations mathematically and understand the relationship between problems in context and mathematical models that may be applied to solve them
- draw diagrams and sketch graphs to help explore mathematical situations and interpret solutions
- make deductions and inferences and draw conclusions by using mathematical reasoning
- interpret solutions and communicate their interpretation effectively in the context of the problem
- read and comprehend mathematical arguments, including justifications of methods and formulae, and communicate their understanding
- read and comprehend articles concerning applications of mathematics and communicate their understanding
- use technology such as calculators and computers effectively, and recognise when such use may be inappropriate.

Subject content

Structure

6. A level further mathematics has a prescribed core and 50% of its overall assessment must address this content. The core content is set out below in sections A to H. For the remaining 50% of the content different options are available. The content of these options is not prescribed and will be defined within the different awarding organisations' specifications. These options could build from the applied content in A level mathematics, they could introduce new applications, they could further extend the pure mathematics content defined below, or they could involve some combination of these.

7. The content of AS further mathematics must introduce new content, build from the AS content of A level mathematics, or a combination of both. It must not overlap with, or depend upon, other A level mathematics content. A minimum of 30% of the assessment of AS further mathematics must address the prescribed core content of the further mathematics A level. Core content that must be included in AS further mathematics is indicated in sections A to C in bold text within square brackets, and this must represent 20% of the overall assessment of AS further mathematics. In addition, a minimum of 10% of the assessment must address other prescribed core content of the further mathematics A level, which can be selected at the discretion of the awarding organisations.

Background knowledge

8. AS and A level further mathematics specifications must build on the skills, knowledge and understanding set out in the GCSE subject content for mathematics and the subject content for A level mathematics. Problem solving, proof and mathematical modelling will be assessed in further mathematics in the context of the wider knowledge which students taking AS/A level further mathematics will have.

Overarching themes

9. A level specifications in further mathematics must require students to demonstrate the following knowledge and skills. These must be applied, along with associated mathematical thinking and understanding, across the whole content of the A level set out below. The knowledge and skills are similar to those specified for A level mathematics, but they will be examined against further mathematics content and contexts.

10. The minimum knowledge and skills required for AS further mathematics are shown in bold text within square brackets.

OT1 Mathematical argument, language and proof

	Knowledge/skill
OT1.1	**[Construct and present mathematical arguments through appropriate use of diagrams; sketching graphs; logical deduction; precise statements involving correct use of symbols and connecting language, including: constant, coefficient, expression, equation, function, identity, index, term, variable]**
OT1.2	**[Understand and use mathematical language and syntax including 'equals', 'identically equals', 'therefore', 'because', 'implies', 'is implied by', 'necessary', 'sufficient', ∴, =, ≡, ≠, ⇒, ⇐ and ⇔]**
OT1.3	**[Understand and use Venn diagrams, language and symbols associated with set theory including 'complement', ∅, ∩, ∪, ∈, ∉, 𝛿]**
OT1.4	**[Understand and use the structure of mathematical proof, proceeding from given assumptions through a series of logical steps to a conclusion; use methods of proof, including proof by deduction, proof by exhaustion]** and proof by contradiction **[Disproof by counter example]** Construct proofs using mathematical induction; contexts include sums of series, divisibility and powers of matrices
OT1.5	Understand and use the definition of a function; domain and range of functions
OT1.6	Comprehend and critique mathematical arguments, proofs and justifications of methods and formulae, including those relating to applications of mathematics

OT2 Problem solving

	Knowledge/skill
OT2.1	**[Recognise the underlying mathematical structure in a situation and simplify and abstract appropriately to enable problems to be solved]**
OT2.2	**[Construct extended arguments to solve problems presented in an unstructured form, including problems in context]**
OT2.3	**[Interpret and communicate solutions in the context of the original problem]**
OT2.4	Understand that many mathematical problems cannot be solved analytically, but numerical methods permit solution to a required level of

	accuracy
OT2.5	**[Evaluate, including by making reasoned estimates, the accuracy or limitations of solutions]**, including those obtained using numerical methods
OT2.6	**[Understand the concept of a problem solving cycle, including specifying the problem, collecting information, processing and representing information and interpreting results, which may identify the need to collect further information, etc.]**
OT2.7	**[Understand, interpret and extract information from diagrams and construct mathematical diagrams to solve problems]**

OT3 Mathematical modelling

	Knowledge/skill
OT3.1	**[Translate a situation in context into a mathematical model, making simplifying assumptions]**
OT3.2	**[Use a mathematical model with suitable inputs to engage with and explore situations (for a given model or a model constructed or selected by the student)]**
OT3.3	**[Interpret the outputs of a mathematical model in the context of the original situation (for a given model or a model constructed or selected by the student)]**
OT3.4	**[Understand that a mathematical model can be refined by considering its outputs and simplifying assumptions; evaluate whether the model is appropriate]**
OT3.5	**[Understand and use modelling assumptions]**

Use of technology

11. The use of technology, in particular mathematical graphing tools and spreadsheets, must permeate the AS/A level further mathematics specifications.

12. Students must not have access to technology with a computer algebra system (CAS) function in examinations, except where the use of CAS is a specific objective within an optional assessment component of further mathematics.

Detailed content statements

13. A level specifications in further mathematics must include the following content, which makes up 50% of the total content of the A level.

14. A minimum of 30% assessment of AS further mathematics must address this core content; this must include the content shown in bold text within square brackets. Assessment of the bold content must represent 20% of the overall assessment of AS further mathematics.

A Complex numbers

	Content
A1	**[Solve any quadratic equation with real coefficients; solve cubic or quartic equations with real coefficients (given sufficient information to deduce at least one root for cubics or at least one complex root or quadratic factor for quartics)]**
A2	**[Add, subtract, multiply and divide complex numbers in the form $x+\mathrm{i}y$ with x and y real]**
A3	**[Understand and use the complex conjugate; know that non-real roots of polynomial equations with real coefficients occur in conjugate pairs]**
A4	**[Use and interpret Argand diagrams]**
A5	**[Evaluate the modulus and argument of a complex number]**
A6	**[Multiply and divide complex numbers in modulus-argument form]**
A7	**[Construct and interpret simple loci in the Argand diagram such as $\lvert z-a\rvert=r$ and $\arg(z-a)=\theta$]**
A8	Understand de Moivre's theorem and use it to find multiple angle formulae and sums of series
A9	Know and use the definition $\mathrm{e}^{i\theta}=\cos\theta+i\sin\theta$ and the form $z=r\mathrm{e}^{i\theta}$
A10	Find the *n* distinct *n*th roots of $r\mathrm{e}^{i\theta}$ for $r\neq 0$ and know that they form the vertices of a regular *n*-gon in the Argand diagram.
A11	Use complex roots of unity to solve algebraic and geometric problems

B Matrices

	Content
B1	**[Add, subtract and multiply conformable matrices; multiply a matrix by a scalar]**
B2	**[Understand and use zero and identity matrices]**
B3	**[Use matrices to represent linear transformations in 2-D; successive transformations; single transformations in 3-D (3-D transformations confined to reflection in one of $x=0$, $y=0$, $z=0$ or**

	rotation about one of the coordinate axes)]
B4	**[Find invariant points and lines for a linear transformation]**
B5	**[Calculate determinants of 2 x 2]** and 3 x 3 matrices and interpret as scale factors, including the effect on orientation
B6	**[Understand and use singular and non-singular matrices; properties of inverse matrices]** **[Calculate and use the inverse of a non-singular 2 x 2]** and use the inverse of a 3 x 3 matrix
B7	Solve three linear simultaneous equations in three variables by use of the inverse matrix

C Further algebra and functions

	Content
C1	**[Understand and use the relationship between roots and coefficients of polynomial equations up to quartic equations]**
C2	**[Form a polynomial equation whose roots are a linear transformation of the roots of a given polynomial equation (of at least cubic degree)]**
C3	Understand and use formulae for the sums of integers, squares and cubes and use these to sum other series
C4	Understand and use the method of differences for summation of series including use of partial fractions
C5	Find the Maclaurin series of a function including the general term
C6	Recognise and use the Maclaurin series for e^x , $\ln(1+x)$, $\sin x$, $\cos x$ and $(1+x)^n$ and know the ranges of values for which they are valid
C7	Calculate errors in sums, differences, products and quotients
C8	Estimate the error in $f(x)$ when there is an error in x using $f(x+h) \approx f(x) + h f'(x)$
C9	Estimate a derivative using the forward and central difference methods and have an empirical understanding of the relative level of accuracy of the two methods

D Further calculus

	Content
D1	Evaluate improper integrals where either the integrand is undefined at a

	value in the range of integration or the range of integration extends to infinity
D2	Derive formulae for and calculate volumes of revolution
D3	Understand and evaluate the mean value of a function
D4	Integrate using partial fractions (extend to quadratic factors $ax^2 + c$ in the denominator)
D5	Differentiate inverse trigonometric functions
D6	Integrate functions of the form $\left(a^2 - x^2\right)^{-\frac{1}{2}}$ and $\left(a^2 + x^2\right)^{-1}$ and be able to choose trigonometric substitutions to integrate associated functions

E Further vectors

	Content
E1	Understand and use the vector and cartesian forms of an equation of a straight line in 3D
E2	Understand and use the vector and cartesian forms of the equation of a plane
E3	Calculate the scalar product and use it in the equations of planes, for calculating the angle between two lines, the angle between two planes and the angle between a line and a plane
E4	Check whether vectors are perpendicular by using the scalar product
E5	Find the intersection of a line and a plane Calculate the perpendicular distance between two lines
E6	Interpret the solution and failure of solution of three simultaneous linear equations geometrically
E7	Calculate the vector product of two vectors including link to 3x3 determinant and understand the properties of the vector product

F Polar coordinates

	Content
F1	Understand and use polar coordinates and be able to convert between polar and cartesian coordinates
F2	Sketch curves with r given as a function of θ ; including use of trigonometric functions from A level Mathematics
F3	Find the area enclosed by a polar curve

G Hyperbolic functions

	Content
G1	Understand the definitions of hyperbolic functions $\sinh x$, $\cosh x$ and $\tanh x$ and be able to sketch their graphs
G2	Differentiate and integrate hyperbolic functions
G3	Understand and be able to use the definitions of the inverse hyperbolic functions and their domains and ranges
G4	Derive and use the logarithmic forms of the inverse hyperbolic functions
G5	Integrate functions of the form $\left(x^2+a^2\right)^{-\frac{1}{2}}$ and $\left(x^2-a^2\right)^{-\frac{1}{2}}$ and be able to choose substitutions to integrate associated functions

H Differential equations

	Content
H1	Find and use an integrating factor to solve differential equations of form $\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{P}(x)y=\mathrm{Q}(x)$ and recognise when it is appropriate to do so
H2	Find both general and particular solutions to differential equations
H3	Use differential equations in modelling in kinematics and in other contexts
H4	Solve differential equations of form $y''+ay'+by=0$ where *a* and *b* are constants by using the auxiliary equation
H5	Solve differential equations of form $y''+ay'+by=\mathrm{f}(x)$ where *a* and *b* are constants by solving the homogeneous case and adding a particular integral to the complementary function (in cases where $\mathrm{f}(x)$ is a polynomial, exponential or trigonometric function)
H6	Understand and use the relationship between the cases when the discriminant of the auxiliary equation is positive, zero and negative and the form of solution of the differential equation
H7	Solve the equation for simple harmonic motion $\ddot{x}=-\omega^2 x$ and relate the solution to the motion
H8	Model damped oscillations using 2nd order differential equations and interpret their solutions
H9	Analyse and interpret models of situations with one independent variable and two dependent variables as a pair of coupled 1st order simultaneous equations and be able to solve them, for example predator-prey models

Department for Education

Any enquiries regarding this publication should be sent to us at www.education.gov.uk/contactus.

This document is available to download at www.gov.uk/government/publications.

Follow us on Twitter: @educationgovuk

Like us on Facebook: www.facebook.com/educationgovuk