The National Strategies
Secondary

Teaching mental mathematics from level 5

Measures and mensuration in geometry

department for
children, schools and families

Teaching mental mathematics from level 5

Measures and mensuration in geometry

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Teaching mental mathematics from level 5: Measures and mensuration in geometry

Measures and mensuration in the context of geometry

Measures are closely linked with **geometry** in the National Curriculum and the learning objectives on the *Framework for secondary mathematics*. Decisions about which aspects to emphasise are often determined by context. For example, constructions can be taught with an emphasis on accurate measurement. However, if we want pupils to understand *why* particular constructions work then geometrical reasoning is the most significant feature. Similarly, the rationale for *mental* work on loci relates to geometry rather than measures. Dynamic geometry programs are valuable tools for developing flexible mental images and providing insights into geometrical properties that, in turn, can help in tackling mensuration problems.

The use of comparative measures can be a stimulus to geometrical thinking. Observing lengths or angles that appear to be equal in size, and trying to establish whether they are necessarily equal or not necessarily equal, is the basis for deriving properties of lines, angles and shapes, including congruence and similarity. Access to software, such as dynamic geometry packages, enables us more easily to establish conjectures about equality of measures, by asking questions such as:

- What changes?
- What stays the same?

The activities described in this supplement build upon and develop activities suggested in *Teaching mental mathematics from level 5: Geometry* (DCSF ref: 00693-2009PDF-EN-01). They may be easily adapted to adjust the level of challenge and keep pupils at the edge of their thinking.

Measures

Estimation

Working with measures can provide an ideal context in which to develop pupils' skills in using and applying new knowledge from different strands of mathematics. For example, as pupils make decisions about the measures in a problem, identifying and obtaining necessary information has real meaning.

Estimation is almost entirely a mental activity. A 'point of reference' is useful as a visual or mental comparison to help in making estimates. For example, knowing and recognising your own height or the area of a football pitch can help in estimating other heights or areas. Pupils should be given opportunities to develop their own collection of 'points of reference'. With practice and experience of estimating they will refine and extend their references and improve their skills.

It is helpful to engage pupils in discussion about the size of the interval within which they are able to place an estimate and whether they have a useful point of reference. It is also helpful for pupils to consider their degree of confidence in their estimates, which will be affected by both of these factors. For example, a pupil may be able to say:

- I am 100 per cent confident that your height is between 1 m and 2 m because you are not unusually tall or small.
- I am 80 per cent confident that your height is between 1.6 m and 1.8 m because I am 1.55 m tall and you are a bit taller than I am.

In interpreting the suggested progression described below it should be recognised that estimation becomes more challenging if it is set within a context or uses units that are unfamiliar.

The *Framework for secondary mathematics* supplement of examples, pages 230 to 231, provides further examples.

Make sensible estimates of a range of measures in relation to everyday situations	length, time
	mass
	area
	capacity, volume
	rates, e.g. *speed*

Rounding and continuity

Pupils should be helped to use their experience of the continuous number line to reinforce the continuous nature of many measures (including some that are often treated as discrete, such as age). A consequence is that measurements given to the nearest whole unit, for example, may be inaccurate by up to one half of the unit in either direction. It is important to take this inaccuracy into account when commenting constructively on the results obtained from calculations using these values.

Appreciate the continuous nature of scales	
Appreciate imprecision of measurement	e.g. *How accurate is the area of a rectangle calculated to be 7.8 m^2 from lengths of 6.5 m and 1.2 m?*

The *Framework for secondary mathematics* supplement of examples, page 231 provides further examples.

Conversion

As pupils develop their skills beyond level 5 they should learn to use units and the symbols for units of measure systematically. It is important to consider the units that are most suitable to the context of a problem. It is also helpful to recognise that the conventions of unit notation can provide insights into the nature of the problem. For example:

- If pupils understand indices and reciprocals and can interpret fractions as division then they can see the structure of compound measures. For example, by interpreting km h^{-1} as km $\times \frac{1}{h}$ and km ÷ h, they can connect the unit notation with the idea that speed is distance divided by time. This kind of connection can help with transformation of formulae.
- Conversion between different systems of measurement is usually undertaken by using approximate equivalence. For example, in converting from miles to metres one might take 1 mile to be approximately 1500 metres. By extension, to express a speed of 20 miles/hour (mph) in metres/hour (m/h) one might mentally calculate that at a speed of 20 miles/hour roughly 30 000 m would be travelled each hour.

Many conversion calculations involve making and using approximations. The same considerations apply to accuracy and rounding, as discussed above.

Know rough metric equivalents of imperial measures in daily use	8 km is approximately 5 miles
Convert one unit to another within a metric system within a non-metric system	 Metric length, mass and capacity e.g. *convert 750 g to kg* Metric area e.g. *convert 62,500 m^2 to hectares* Metric volume e.g. *convert 5.5 cm^3 to mm^3* Time e.g. *approximately how many more minutes there are in January than February*
Between two systems	Currency e.g. *convert £12.67 to euros*
Understand and use compound measures	Convert one rate to another e.g. *convert 30 lb per square foot to kg/m^2*

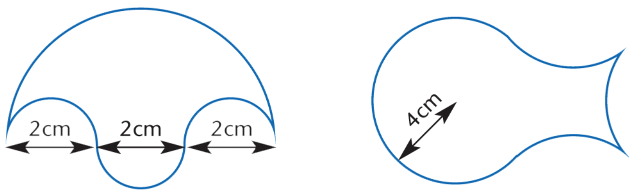
The *Framework for secondary mathematics* supplement of examples, pages 228 to 229 provides further examples.

Mensuration

It is worth considering carefully how to deal with the formulae that are used in mensuration. Pupils are more likely to use and apply a formula with confidence if they can see how it links with the properties of the shape. This understanding may be triggered as pupils work together to deduce formulae and explain their features. In many cases the deduction will develop from knowledge of a more familiar shape. For example, pupils can extend knowledge about rectangles to derive a formula for the area of a parallelogram.

When a particular collection of formulae is established then pupils need practice in using the formulae to solve problems. There are several stages to this problem-solving process and the early and later stages are often given less attention than the central calculation.

- The early stages involve making decisions about which properties are significant and which formulae are appropriate. This is a mental stage and pupils benefit from being able to discuss the options. This can be supported by, for example, sorting and classifying questions into types.
- The later stages also involve decision-making, this time about checking the units and appropriate degrees of accuracy. Pupils should consider questions such as: 'How will the accuracy of input values affect the resulting solutions?'
- Pupils should also be encouraged to consider the links between a formula and the sort of units they might expect for the answer, and to check that they are consistent. For example, the formula $A = 2\pi r$ cannot be correct because A (area) requires square units and $2\pi r$ contains only one linear unit.

Perimeter	
rectangle, triangle, compound shapes, circle, circular arcs	

The *Framework for secondary mathematics* supplement of examples, pages 234 to 236, provides further examples.

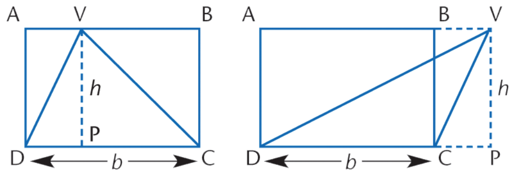
Area

Rectangle:

compound rectangles (bits in and bits out);

'adapted' rectangle (parallelogram,

trapezium, right-angled triangle)

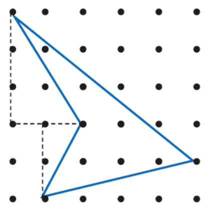


Triangle:

compound (bits in and bits out);

'adapted' triangle (rhombus, kite, trapezium,

parallelogram, regular hexagon)

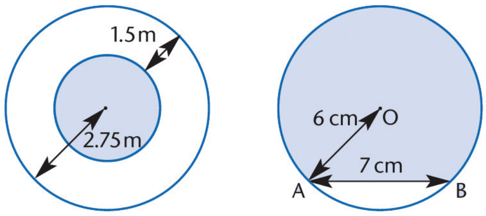


Circle:

compound (bits in and bits out);

'adapted' circle (semicircle, quadrant,

sector, segment)



The *Framework for secondary mathematics* supplement of examples, pages 234 to 237, provides further examples.

Volume and surface area

Identify and use the measurements necessary to calculate the volume and surface area of:

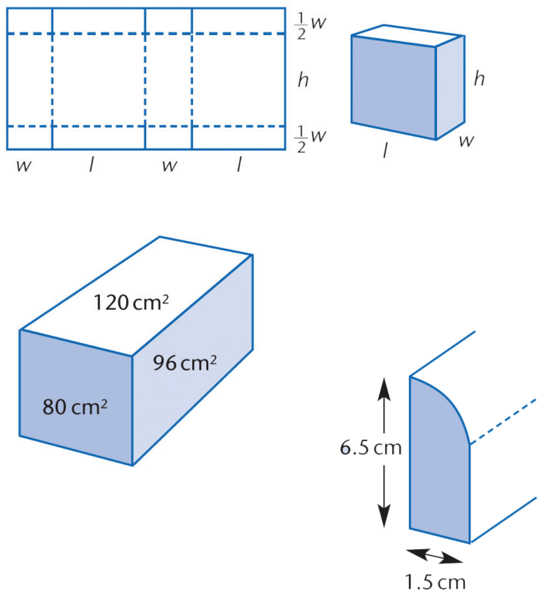
cube

cuboid

right prisms including cylinder

cone and sphere

compound 3-D shapes



The *Framework for secondary mathematics* supplement of examples, pages 238 to 241, provides further examples.

Compound measure

The calculation stages of a problem involving compound measures can cause difficulties if pupils do not give some time to the mental stages at the start. At the outset it is important to establish a clear rationale for the units in the question and how these link together. This clarity can help to inform the various stages of any calculation and the final form of the solution. To develop this habit of mind, it can be worthwhile concentrating only on this 'entry stage' for a collection of problems. The solution may not be completed but pupils should verbalise the units and identify the link between these units and the nature of the measures in the calculation and solution. For example, they might identify the stages and associated units in working out the cost of petrol for a journey of 140 miles if the minibus averages 11 miles to the litre and petrol costs 90 pence per litre.

It is through a more thorough understanding of the connections between all the units in a mensuration problem that pupils will be better able to determine the dimension of formulae from inspecting the nature of the variables involved. This understanding will also reinforce accuracy when pupils are required to change the subject of a mensuration formula, for example, amount of petrol = rate of consumption × distance travelled.

Understand and use compound measure, for example, speed, density or pressure to solve problems
Petrol consumption 60 litres per km
Speed of Pluto's orbit 1.06×10^4 mph
Density of gold 19.3 g/cm^3
Atmospheric pressure 1.013×10^5 N/m^2

The *Framework for secondary mathematics* supplement of examples, page 233 and page 21, provides further examples.

Activities

Paper-folding, using a piece of A4 paper, will produce an interesting set of ratios because of the particular properties of the metric paper sizes. Fold an A4 page in half (bisecting the longer sides) just once. Confirm visually, by comparing a folded sheet to a full sheet, that the folded piece seems to be a similar shape to the original sheet. Fold again and make the same visual comparison. Pupils could now be asked to discuss how they would investigate this further. The mental part of this task is at the planning stage, where pupils will benefit from time to speculate and discuss before they are distracted by the detail of measuring or calculating.

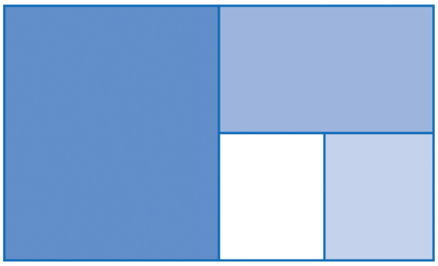
- If the shapes are similar, what is the ratio of enlargement? Why?
- What is the ratio of length to width for each sheet? Do you need to find this by measuring? How might rounding affect this?
- Does this look as though it would continue for other larger and smaller sizes of metric paper?
- What happens if you begin with a sheet of non-metric paper?
- What happens if you begin with a square sheet of paper?

For a visual image of the ratios within and between similar rectangles, see the 'Photographic enlargements' available on the Framework for secondary mathematics

Halving an area by shading also gives fascinating results. Start with a rectangular piece of paper in landscape format and draw a line down the middle to divide it in half. Shade the area to the left of the line. Note its area as a fraction of the whole sheet.

Working clockwise, turn the paper so that you have a landscape view of the unshaded area; draw a vertical line to divide it in half. Shade the left-hand half and note its area as a fraction of the whole.

Continue to work round clockwise, following the same procedure and noting the new area shaded at each stage. Keep going!



In theory you could keep going for ever.

- What sequence of fractions is generated as you progress?
- If you begin with an A0 sheet of paper (area 1 square metre) can you say anything about the sum of this sequence of fractions?

The same context can be used to consider sequences of length as an alternative to area.

- What is the total distance walked by an ant starting at the bottom left and walking clockwise round two adjacent sides of each new rectangle?

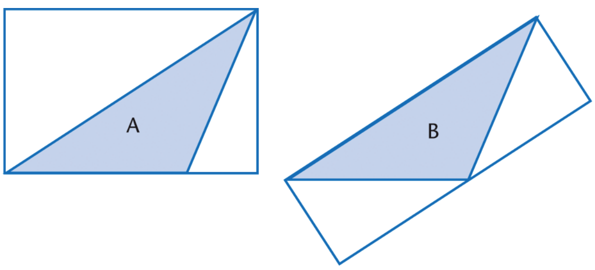
Pupils should note that finding relationships between length and area often involves reasoning through the properties of the shapes in question and does not always involve measuring. In fact, it is worth noting that the rounding that results from measuring can often obscure more general results.

Area and perimeter – always, sometimes or never true? is an activity in which pupils are given statements and asked which are always true, which are sometimes true and which are never true. They are encouraged to discuss the problem in order to give reasons for their responses. This type of activity is useful to encourage pupils to explore relationships between measures and to check both the reasonableness and the generality of results. Sample statements include:

- The numerical value of the area of a rectangle is greater than the numerical value of its perimeter.
- The numerical value of the perimeter of a concave quadrilateral is greater than the numerical value of its area.

These measurements are, of course, in different units since only one is linear. Nonetheless, an exploration of the relationship between the numerical values of these measurements provides insights into their properties and is a useful precursor to maximisation problems.

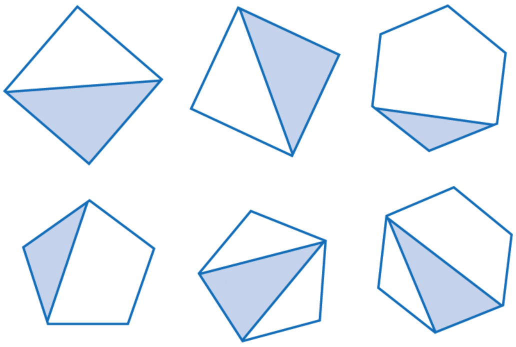
- The area of a triangle is half of the area of the smallest rectangle that encloses it.



Prompt pupils to look for a counter-example if they are unsure about this statement. The case that causes difficulty here is that of the obtuse-angled triangle. Comparing the rectangles enclosing the same triangle in examples A and B above should expose erroneous thinking.

- The area of a rectangle is equal to half of the product of its diagonals.
- Halving the diameter of a circle halves its area.

Triangles in polygons - true or false? requires pupils to assess whether given statements are *necessarily true or necessarily untrue*. The statements refer to diagrams showing triangles within pairs of congruent regular polygons. The focus is on general relationships within regular polygons and so *sometimes true* is not an option.



Ask the pupils to consider statements such as:

- The triangles in the squares have the same area.
- The triangles in the regular pentagons are equal in area.

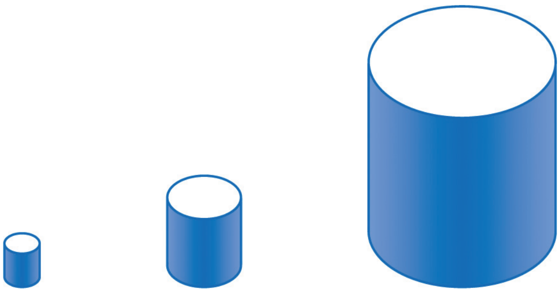
The squares should be fairly straightforward but many pupils are convinced that the triangles in the pentagons are equal in area, perhaps confusing the image with others they remember. Allow time for pupils to discuss the conflict that they may experience at this stage. In all cases, use of dynamic geometry software can help pupils to test the truth or otherwise of statements.

- The areas of the triangles drawn in the regular hexagons are in the ratio 1 : 2.
- In a regular octagon, there are four different triangles that can be drawn from one side to a vertex, all with different areas.

The different triangles that can be drawn in a regular octagon are unequal in area but only three such triangles can be drawn, as the remaining vertex of each triangle must be one of six points that are arranged symmetrically.

- If all the polygons have the same side length, the triangles drawn in the squares would have the largest area.

Similar solids, such as the three cylinders shown in the diagram overleaf, have dimensions in proportion. In this case their diameters are in the ratio 1 : 2 : 5. The principle of surface areas or volumes being proportional to the squares or cubes of the linear dimensions can be challenging for pupils to apply. Approaching the problem by planning the strategy before calculating can force attention on the structure of the problem and its general application.



Ask pupils to design a table (see below) to show the ratios of length, surface area and volume. They should then practise stating the single multiplier that would take them from one cell to an adjacent cell in the table. They should move around the table in as many ways as possible.

	small	medium	large
diameter (length)	1	2	5 ↑
surface area	1	4 →	25
volume	1	8	125

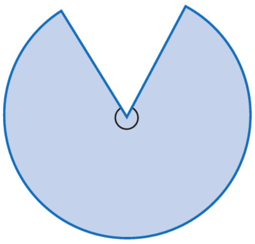
For example, the multiplier for the horizontal arrow shown is $\frac{25}{4}$ and the multiplier for the vertical arrow shown is $\frac{5}{25}$ or $\frac{1}{5}$.

When pupils are confidently giving such multipliers for any pair of cells they can apply their strategy to problems such as:

If the middle cylinder has a volume of 160 cm^3 and the large cylinder has a surface area of 1000 cm^2, calculate the two remaining surface areas and volumes.

For further examples on similarity and enlargements, see the *Framework for secondary mathematics* supplement of examples, pages 213 to 217.

Max-cone challenges pupils to consider the angle of the sector that must be cut from a circle to produce a cone of maximum volume.



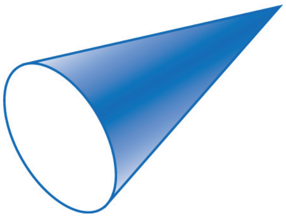
Avoid measurement or calculation and focus on presenting a chain of reasoning that describes the nature of the shape as the angle increases. Pupils should aim to produce descriptions, using accurate and appropriate vocabulary, that help other pupils to visualise the changing solid. They should describe the balance between the angle of the sector, the area of the base circle of the cone, the height of the cone and its volume.

Following this description it is likely that pupils will have formed an hypothesis about the point of maximum volume. This is the stage at which the mental work is supported by trial and improvement, using a spreadsheet. The resulting volumes could be graphed against the angles chosen.

Extended exploration around problems of this nature can help pupils avoid the mistake of rushing too soon to start calculating before a clear understanding and strategy have emerged.

Further examples on areas of circles and sectors are available in the *Framework for secondary mathematics*, pages 235 and 237.

Rolling cones can be carefully placed on their sides and rolled round in a circle so that they return to their exact start point.

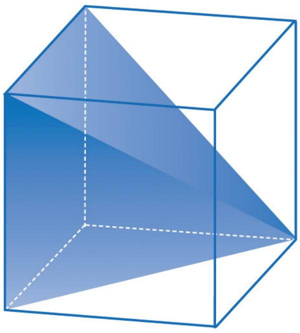


Imagining this movement should encourage pupils to focus mentally on the properties of the shape and the relationship between various measurements that might be considered. For example:

- If the slant height of a right circular cone is double the diameter of its base, how many times does the cone rotate about its axis in making this journey?
- Is it possible to generalise the result by considering similar problems with different links between diameter and slant height?

Pyramid inside a cube encourages pupils to look for shapes within shapes. This helps them to develop and sustain the imagery required to tackle mensuration problems. Some powerful concepts can be accessed by pupils exploring the relationships between different 3-D shapes.

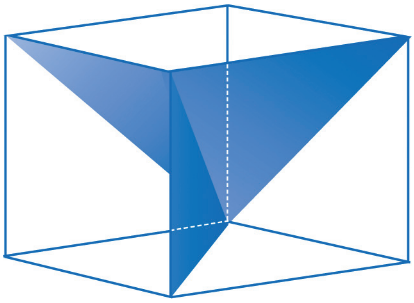
Show a diagram of a skew pyramid inside a cube.



Ask:

- How many other identical pyramids would fit in the cube?
- How can you be sure? What could you do to convince yourself?
- What does this tell you about the relationship between the volume of the pyramid and the volume of the cube?

Now show a diagram of 'open' skew pyramids built inside a cuboid.



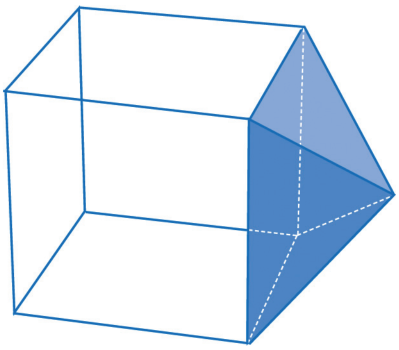
Ask:

- Would the same relationship hold for this diagram as held for the cube above?
- Why? Explain your reasoning.

Inside and out involves visualising shapes within shapes. Ask pupils to begin by imagining a net for a cube, laid out flat on a table. They should imagine building congruent square-based pyramids on each face and then folding up the net to form a cube with the pyramids inside. Tell pupils that it is possible to design the six pyramids so that they meet **inside** at the centre of the cube and ask them to picture this. Allow a few minutes for pupils to discuss their images and decide whether they are convinced about the inside shape. Then ask:

- If the edge of the cube is 6 cm, how would you work out:
  - the lengths of the slant edge of each pyramid;
  - the angle each slant face of the pyramid makes with a face of the cube?

A challenging extension of this idea would be to make or design a model to show the pyramids on the *outside* of each face of the cube and to analyse the 3-D shape that results.



In this diagram, the *external pyramid* is shown on only one of the faces.

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