Teaching mental mathematics from level 5

Geometry

Department for Education

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Introduction

What is mental mathematics?

Almost all of mathematics could be described as 'mental' in the sense that engaging in a mathematical task involves thinking. Thus every mathematical problem a pupil tackles must involve several stages of mental mathematics. Pupils actively involved in mental mathematics might be engaged in any combination of:

interpreting visualising analysing synthesising explaining hypothesising inferring deducing judging justifying making decisions

These ideas are prevalent throughout mathematics and underpin mathematical processes and applications.

If the definition is so wide ranging, how have we produced a few brief booklets with this title? The answer is that we have been very selective! The 'mental mathematics' supported through the teaching approaches described in these booklets is aimed at a subset of mental mathematics in its broadest sense. We have chosen a few key areas likely to influence pupils' progress beyond level 5. These selections have been informed by recent annual standards reports from Qualifications and Curriculum Development Agency (QCDA) and the experience of teachers and consultants. The initial ideas have also been supported by classroom trials.

How do I help pupils to improve the way they process mathematics mentally?

Individual pupils will be at different stages but all pupils develop some strategies for processing mathematical ideas in their heads. Many of the activities suggested in these booklets increase the opportunities for pupils to learn from one another by setting them to work collaboratively on tasks that require them to talk. Often pupils develop and enhance their understanding after they have tried to express their thoughts aloud. It is as if they hear and recognise inconsistencies when they have to verbalise their ideas.

Equally, new connections can be made in a pupil's 'mental map' when, at a crucial thinking point, they hear a different slant on an idea. A more discursive way of working often allows pupils to express a deeper and richer level of understanding of underlying concepts that may otherwise not be available to them. In this way pupils may:

- reach a greater facility level with pre-learned skills, for example, becoming able to solve simple linear equations mentally
- achieve a leap in understanding that helps to complete 'the big picture', for example, seeing how the elements of a function describing the position-to-term relationship in a sequence are generated from elements in the context of the sequence itself.

The activities are designed to engage pupils in group work and mathematical talk.

Is mental mathematics just about the starter to the lesson?

Developing mental processes is not simply about keeping some skills sharp and automating processes through practice. The activities described in this booklet support the main part of the lesson. Developing a mental map of a mathematical concept helps pupils to begin to see connections and use them to help solve problems. Developing the ability to think clearly in this way takes time. Once in place, some aspects of mental mathematics can be incorporated into the beginning of lessons as a stimulating precursor to developing that topic further.

The activities are intended to support the main part of the lesson.

Is mental mathematics just about performance in mental tests?

Using these materials will help pupils to perform more successfully in tests, but the aim is more ambitious than that. Developing more effective mental strategies for processing mathematical ideas will impact on pupils' progress in mathematics and their confidence to apply their skills to solve problems.

Secondary teachers recognise the importance of pupils' mathematical thinking and application, but few have a range of strategies to support its development. The expectations described, and the activities suggested in the accompanying mental mathematics resources, aim to create a level of challenge that will take pupils further in their thinking and understanding. These materials should provide the chance for pupils to interact in such a way that they learn from each other's thinking, successes and misconceptions and thereby become increasingly confident and independent learners.

Pupils need to transfer mathematics confidently and apply it whenever they need to use it. This needs to be taught. Most commonly, pupils will use mental mathematics in solving problems as they occur in their lives, in other areas of their studies and as they prepare for the world of work. To support pupils in doing this, teachers will frequently need to set both large and small mathematical problems in real, purposeful and relevant contexts. Pupils will need to solve increasingly complex and unfamiliar problems using mathematics, apply more demanding mathematical procedures during their analysis and do so with increasing independence. These materials support teachers in planning a structured and progressive approach to do this. If learning is planned with mental mathematics as a significant element, pupils will develop increasing confidence in applying mathematics.

Improving mental mathematics will improve pupils' confidence to apply what they know.

Can mental mathematics involve paper and pencil?

Mathematical thinking involves drawing on our understanding of a particular concept, making connections with related concepts and previous problems and selecting a strategy accordingly. Some of these decisions and the subsequent steps in achieving solutions are committed to paper and some are not. When solving problems, some of the recording becomes part of the final solution and some will be disposable jottings.

Many of the activities involve some recording to stimulate thinking and talking. Where possible, such recording should be made on large sheets of paper or whiteboards. This enables pupils, whether working as a whole class or in pairs or small groups, to share ideas. Such sharing allows them to see how other pupils are interpreting and understanding some of the big mathematical ideas. Other resources such as diagrams, graphs, cards, graphing calculators and ICT software are used in the activities. Many of these are reusable and, once developed in the main part of a lesson, can be used more briefly as a starter on other occasions.

Progress may not appear as written output. Gather evidence during group work by taking notes as you listen in on group discussions. Feed these notes into the plenary and use them in future planning.

The materials

Each attainment target in mathematics is addressed through its own booklet, divided into separate topic areas. For each topic, there is a progression chart that illustrates expectations for mental processes, broadly from level 5 to level 8. Mathematical ideas and pupils' learning are not simple to describe, nor do they develop in a linear fashion. These are not rigid hierarchies and the degree of demand will be influenced by the context in which they occur and, particularly for the number topics, by the specific numbers involved. For this reason ideas from one chart have to interconnect with those in another. The aim is that the charts will help teachers to adjust the pitch of the activities that are described on subsequent pages.

There are many National Strategies materials which reinforce and extend these ideas but, to ensure that these booklets are straightforward and easy for teachers to use, cross-referencing has been kept to a minimum. The most frequent referencing throughout the booklet is to the *Supplement of examples*, which is now connected to the Framework for Secondary Mathematics (**www.standards.dcsf.gov.uk/nationalstrategies**). The page numbers of the original supplement have been retained and the examples can be downloaded as a complete document or in smaller sets from the related learning objectives.

Teaching mental mathematics from level 5: Geometry

The topics covered in this chapter are:

- geometric reasoning: lines, angles and shapes
- using symmetries, reflections, rotations and translations
- enlargement and similarity
- constructions and loci
- working in three dimensions.

These are selected from the geometry section of the learning objectives on the *Framework for secondary mathematics*. 'Working in three dimensions' is not distinct from 'Geometric reasoning' but is a challenging aspect within it and therefore has been covered in a section of its own. The topics also include some of the aspects of geometry that have been reported as having implications for teaching and learning from the Key Stage 3 tests. For example, to help pupils improve their performance, teachers should:

- give pupils more opportunities to work with and visualise shapes in unfamiliar orientations
- encourage pupils to use the mathematical language associated with a topic
- encourage pupils to visualise or reason about geometrical problems in two and three dimensions
- use problems involving the recall of angle properties to encourage pupils working at level 6 and above to construct chains of reasoning.

The tasks described in this chapter engage teachers and pupils in reasoning through language and images. Pupils working collaboratively in this way have a greater chance of understanding the 'big picture' as they take advantage of opportunities to reprocess their thinking and make new connections.

In the following sections several strategies are used repeatedly and are worth general consideration.

- **Classifying** is a task well suited to the thinking processes that everyone uses naturally to organise information and ideas. A typical classification task may involve a card sort. Pupils work together to sort cards into groups with common characteristics that establish criteria for classification. Being asked to consider and justify their criteria helps pupils to develop their skills and understanding. The key part of designing a good classification task is the initial choice of cards that will provide a sufficiently high challenge. A common mistake when running a classification task is to intervene too soon and over-direct the pupils.
- **Matching** different forms of representation often involves carefully selected cards and a lesson design similar to that for classifying. In this instance pupils are asked to match cards that are equivalent in some way. This kind of activity can give pupils important mental images, at the same time offering the chance to confront misconceptions.
- **Visualisations** as an aspect of mathematical activity are recognised in the National Curriculum as important in developing pupils' skills in geometric reasoning. To develop these skills fully, pupils need to have regular opportunities to extend their mental imagery through structured visualisation tasks. These can be used to help pupils construct and control mind pictures, see geometrical relationships and develop the language to describe what they see. To begin with they need only be quite short, perhaps no more than 2 or 3 minutes. The idea is to ask the class to imagine a simple picture of lines or shapes, building up or altering the picture in various ways and asking them to notice features, what changes and what stays the same. This is usually followed by a whole-class

discussion to reconstruct what they saw, probably taking quite a bit longer than the visualisation itself. There is a discipline to controlling and working with mind pictures, so resist the temptation to draw pictures until later in the reconstruction – when clarity is needed but words fail!

- **Always, sometimes or never true?** is a task in which pupils decide whether a statement is always true, sometimes true or never true. It is crucial that pupils explain the reasons for their responses. Statements selected for this task have most value when they are likely to be challenging to pupils' concept development: when they encompass likely misconceptions. For example, a statement about properties of shapes could challenge perceptions of shapes in prototypical orientation; some pupils see a square in an unfamiliar orientation as a 'diamond'. In each section, start with some simple statements that clearly fall into one of the three categories so that pupils can become familiar with the level of reasoning that is expected. The task can be organised in various ways, including group or paired work, with statements for pupils to discuss written on cards, or for whole-class discussion, using a computer projection system or interactive whiteboard and 'drag and drop' techniques.
- **Wise words** is a versatile task suitable for developing understanding of most visual forms. Pupils work in pairs, with a set of up to eight cards or objects. All pairs use identical sets. Pairs compose two statements to describe a chosen shape or object. Another pair must try to identify the object within the set, from the statements. The two statements should focus on different key features; for example, one statement could be about *either* sides *or* angles and the other statement could be about diagonals. Each statement must use only one of the key words given (for example, *parallel, equal, bisect, perpendicular, opposite, adjacent*). Each pair passes their statements to another pair, for them to work out which item is being described. The second pair can write one question with a 'yes/no' answer. Then, after getting a response to this, they must identify the selected object or image on the card. The choice of object or image on the cards, the number of items and the key words make this a rich and adaptable activity, engaging pupils in discussion and forcing them to consider the precision of the language they are using.

Further support for the teaching of geometry is available from:

- The *Framework for secondary mathematics*
- *Geometric images*, ATM (ISBN 0 900095 36 9). Written in 1982 but just as relevant today, this small booklet provides a rich source of ideas for developing mental imagery in geometry. Also available from ATM publications, *Geometry games* (ISBN 1 898611 38 6) contains a range of games that enable pupils to develop their understanding of the geometric properties of 2-D shapes.
- Support through ICT is available from the website of the Mathematical Association (**www.m-a.org.uk**). Resources include interactive downloads based on the geometry examples shown in the *Framework for secondary mathematics* and are available free of charge.

Geometric reasoning: lines, angles and shapes

While the whole of the contents of this booklet could be described as geometrical reasoning, the National Curriculum and the *Framework for secondary mathematics* both identify work on lines, angles, triangles and other rectilinear shapes and properties of circles and 3-D shapes under this heading. Much of this work in written mathematics relates to traditional areas of study. In terms of ideas and understanding, however, links to other aspects of shape, space and measures are very evident. Work on similarity (in the geometric reasoning section), for example, arises out of enlargements (transformations) and contributes significantly towards pupils' understanding of trigonometry (measures and mensuration). Teachers often build upon pupils' experiences of using *repeat* in Logo to construct regular polygons (constructions and loci) to establish a means of calculating exterior angles (geometric reasoning).

Mental imagery and visualisation are key elements in developing pupils' capacity to solve problems, especially those in geometry. Pupils need to become practised in constructing chains of reasoning for their hypotheses about properties of shapes or sizes of angles. They should be taught to distinguish between the conventions of geometry, the definitions or givens and the properties that are derived from them. Developing understanding and using chains of reasoning, as part of collaborative activities in the classroom, increase the likelihood that this reasoning will become a routine part of geometric thinking and thus allow greater access to more challenging problems. It is common to deal with conventions rather too quickly but, as geometry is about linking ideas and communication, these are the necessary building blocks of written proof and justification.

This section deals with the processes required to solve geometric problems mentally. The intention is to develop a mental facility to recognise the geometric features in a shape. These might be inherent in the shape or introduced to the shape by construction. Recognising these features is helpful in solving problems. Time spent developing these skills at an early stage can help to increase pupils' confidence and competence when later problems become quite complex.

Recognise and explain properties of given shapes to calculate angles and/or derive other properties:
knowing the angle sum at a point, on a straight line and in a triangle
using angle properties of equilateral, isosceles and right-angled triangles
using angle properties of intersecting and parallel and perpendicular lines
knowing and using geometric properties of triangles and quadrilaterals in classifying different types of quadrilateral
knowing angle and symmetry properties of polygons and using them to solve problems
explaining and justifying geometric inferences and deductions, using mathematical reasoning
knowing and understanding conditions for congruent triangles to find angles and in geometric proofs
identifying and using properties of circles to solve problems

The Framework for secondary mathematics supplement of examples, pages 178 to 191, provides contexts in which pupils should develop mental processes in geometric reasoning.

Activities

Classifying cards could include:

- diagrams of shapes
- names of shapes
- properties of shapes, for example, *angles*, *sides*, *symmetries*, *diagonals*.

This task will involve pupils, working in pairs or small groups, in negotiating the meaning of geometric terms. Although written output may be minimal, important diagnostic feedback can be obtained by listening and observing as pupils, quite literally, lay their thinking out on the tables.

Sorting properties could be a follow-up to the classifying task. Examples of such tasks can be found in the photocopiable resource from ATM, *Geometry games* (ISBN 1 898611 38 6). A simple activity from this collection is called *Sort the properties*. In this game, pupils work in pairs or groups of four to sort two types of card showing names and properties of quadrilaterals (see resource 1, *Sort the properties*, pages 34 and 35). They arrange the cards in pairs, so that each pair makes a true statement. Once this has been done you can explore whether the solution is unique or not. Further discussion may involve a notion of 'minimum' collections that, by themselves, are sufficient to define a shape. This can lead to some unlikely definitions, for example, 'a quadrilateral in which diagonals bisect each other'. Pairs could attempt to compile different minimum collections of cards and present them to another pair asking, 'Is this enough? Is it too much?' The checking pair should look at two criteria.

- Is it possible to use these properties to draw any shape other than the target shape?
- Is it possible to remove a property card and still only be able to draw the target shape?

For ideas for card sets for these activities, see the Year 8 expectations in *The Framework for secondary mathematics* supplement of examples, page 187.

For a similar activity to use with an interactive whiteboard or computer screen projection system, see the ideas for teaching mathematics through ICT on the *Framework for secondary mathematics*

Geometry games (ISBN 1 898611 38 6) from ATM includes a wealth of games aimed at developing geometrical reasoning (**www.atm.org.uk**).

Sharing the image involves one pupil interrogating another in order to identify a mystery shape. The mystery shape is visible only to the answering pupil, as a sketch or object, on a geoboard or on the screen of a computer running dynamic geometry software. The interrogating pupil must identify the shape by asking questions that require only 'yes/no' answers, using precise language relating to the geometrical properties of the shape. The aim is to identify shapes by asking as few questions as possible. This task helps pupils to communicate accurately and concisely, to develop their ability to transmit and receive information, to enhance their powers of mental imagery and to develop mathematical vocabulary.

Always, sometimes or never true? generates fruitful discussion. Refer to the geometric properties of quadrilaterals outlined on page 187 in the Year 8 outcomes of the *Framework for secondary mathematics*:. From these properties, compose a set of statements that pupils can classify as 'always true', 'sometimes true' or 'never true'. For example:

- The diagonals of a parallelogram bisect each other.
- A kite has rotational symmetry of order 2.
- An isosceles trapezium has unequal diagonals.

Pupils, working in pairs on this activity, can usefully share their solutions with another pair, who may have reached different conclusions to their own. They must use geometrical argument to justify their conclusions. Plenary prompts might be:

- How do you know that the diagonals of a parallelogram *always* bisect each other?
- Is it possible for a kite *sometimes* to have rotational symmetry of order 2? In this case, how would you rename the shape?
- What property of an isosceles trapezium means it will *never* have unequal diagonals?

An alternative to this approach is to ask pupils to devise statements that fit the criteria *always*, *sometimes* or *never true* in the context of a particular type of quadrilateral.

Rather than using the terms *always*, *sometimes* or *never true*, Year 9 pupils working on such activities could be introduced to the language of proof: *necessarily true*, *undetermined* and *necessarily not true***.**

Visualising a square dissection sets pupils the task of visualising a square and creating 'in their mind's eye' a right-angled triangle and a trapezium, formed by a single cut from a vertex to the mid-point of an opposite side. After agreeing this image, pupils could attempt further tasks that involve:

- creating a mental picture of the different shapes made by joining equal sides of the right-angled triangle and trapezium
- naming and describing (not drawing) the shapes to check that all possibilities have been found
- producing a chain of reasoning to explain how they know they have found all the shapes possible.

A variety of short visualisation activities is available in the *Framework for secondary mathematics* supplement of examples, pages 184 to 185.

Designing quadrilaterals is an activity in which pupils are asked to choose any quadrilateral with a specified property. They consider changes they could make to the shape while maintaining the stated property. They state what new shape is produced and why. For example, 'diagonals at right angles' could link to a square as a start point. The square could be modified by making the two intersecting diagonals longer or shorter rather than the same length. So pupils would nominate *kite*, *rhombus* and *square*. They could then pick one image and describe it to their neighbour. They should focus on particular combinations of properties and how the collections of these produce new shapes, for example, diagonals equal in length with both/one/neither bisected; diagonals unequal in length with both/one/neither bisected. In all cases it is important that pupils try to generate mental images first but they might usefully support these in subsequent discussions by means of dynamic geometry software.

Further opportunities for pupils to explore the links between properties of diagonals of a quadrilateral and the quadrilateral itself can be found in the QCDA booklet, *Developing reasoning through algebra and geometry*, page 13: Rich activity, local deduction, and page 17: Is it still true? (ISBN 1 85838 550 4) See **www.qcda.gov.uk**.

Wise words requires pupils to describe succinctly what they see. They will need a set of eight cards, each showing a different quadrilateral. Pupils, working in pairs, choose one image and make two statements about it. One statement should be about *either* sides *or* angles and the other statement should be about the diagonals. The information given should be sufficient to define the generic quadrilateral; for example, *opposite sides are parallel* and *diagonals bisect each other* would not be sufficient to define a square. Each statement must use only one key word (*parallel, equal, bisect, perpendicular, supplementary)* and must not reveal the name of the quadrilateral shown on the card. The other pair tries, by asking as few 'yes/no' questions as possible, to identify the shape.

Using a **Two-way table,** such as the one below, can engage pupils in discussing geometrical properties of quadrilaterals and other shapes. Pupils are asked to write the names of appropriate quadrilaterals in the spaces and to consider whether any spaces will remain empty and, if so, why.

		Number of pairs of parallel sides		
		0	1	2
Number of pairs of equal sides	0			
	1			
	2			

Other properties for sorting include:

- number of right angles
- number of lines of symmetry
- number of sides
- number of sides of different length
- order of rotational symmetry.

Using symmetries, reflections, rotations and translations

Symmetry provides a natural and fascinating source of mathematical imagery. Its application in mathematical reasoning can save time and effort and provides an example of the power of mathematics. The dynamic nature of transformations makes them a rich source for exploring mental imagery in the solution of problems. These aspects of mathematics are particularly fruitful for kinaesthetic learners and those whose thinking is supported by visual pattern.

Classroom experience suggests that pupils' understanding of symmetries, reflections, rotations and translations can be developed through a range of practical and mental activities, for example:

- developing mental imagery, perhaps using tracing paper overlays to build up geometrical diagrams
- using practical materials such as tangrams to make particular shapes, or mathematical tiles to create tessellations
- using dynamic geometry software to create and transform shapes and patterns.

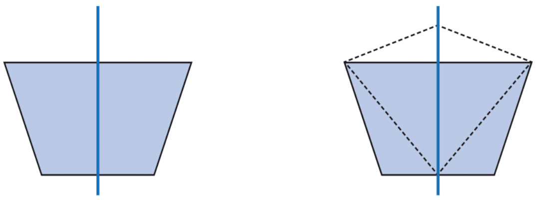
Such experiences help pupils to develop their mental imagery as a vital tool in identifying how shapes are constructed, in recognising their common features and in building strategies that can help them to solve geometric problems.

Understand and use symmetries and transformations of 2-D shapes:
recognising and visualising reflections, translations and line symmetries of shapes
recognising and visualising rotations and rotational symmetries of shapes
deducing geometrical and angle relationships from tessellations and other patterns
recognising and visualising combinations of two or more transformations
finding transformations to fit certain criteria
understanding and being able to explain a proof, for example, of Pythagoras' theorem, using symmetries and transformations

The *Framework for secondary mathematics* supplement of examples, pages 202 to 215, provides contexts in which pupils should develop mental processes in symmetry and transformations.

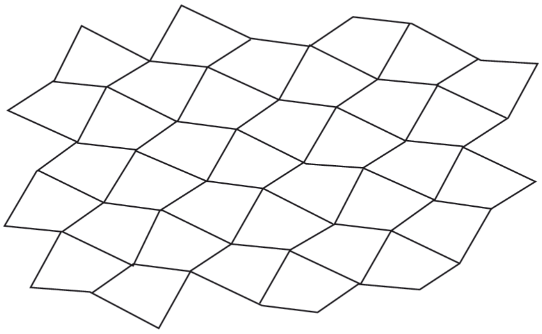
Activities

Designing quadrilaterals requires pupils to choose any quadrilateral that has a specified symmetry. They consider changes they could make to the shape while maintaining the stated property. They state what new shape is produced, and why. For example, 'has exactly one line of symmetry' could link to an isosceles trapezium as a start point. This shape could be modified by reducing the length of one of the parallel sides. As the length reduces towards zero pupils might notice that they are in danger of 'losing a side' but could create a new side by allowing the other parallel side to split at the point where it is crossed by the line of symmetry. They would then create a kite.



Thus they identify the range of quadrilaterals to which a given property applies. A common property of a group of shapes can stimulate thinking about why only these shapes appear in this group. The rigour of pupils' reasoning develops with practice. To practise collaborative reasoning they must feel confident to take risks in their thinking, trying to make connections rather than merely reinforcing routines. For example, if the property is 'two lines of symmetry' pupils might explain that: 'Either the sides of the quadrilateral have to be at right angles to the lines of symmetry, or both the lines of symmetry go through a vertex.'

See the *Framework for secondary mathematics* supplement of examples, pages 185, 209 and 211.



You can access a dynamic version of this image at www.chartwellyorke.com/mentalgeometry/quadtess.html. This and subsequent links are provided purely as examples and their inclusion should not be interpreted as a recommendation or endorsement, either by the DCSF or the Secondary National Strategy, of the software used to generate the materials or the suppliers of such software or materials.

Transformations and tessellations can be developed through the mental imagery involved in the use of tiles or dynamic geometry software. Using the two media may help to consolidate different aspects of the same concept. This is more than just pattern and play. Although pupils' thinking is supported by 'props' the requirement for argument and justification can challenge mental processes, thereby involving much conjecture; for example, 'If the angles were different then…. If an angle was bigger than 180….'

Use dynamic geometry software to engage pupils in the design of a tessellation. Ask:

- How can you make one triangle tessellate?
- Would this also work for quadrilaterals?
- Can you explain why the tessellation works?
- Can you prove why it works?

Use **tiles** of regular polygons, such as ATM mathematical activity tiles. Ask:

- Which regular polygons tessellate by themselves?

As pupils consider these *regular tessellations* their attention could be focused on the interior angles of each regular polygon by asking:

- Why do you think the tessellation works?

Pupils may naturally move on to consider *semi-regular tessellations*, or may need prompting:

- How could you make two regular polygons tessellate?

Again observation about interior angles is important:

- Looking at the other polygons composed by the shapes in a tessellation, what can you say about the interior angles?
- Looking at all the regular polygons available, how many semi-regular tessellations do you think there are? Produce a chain of reasoning to show how you know.

See the *Framework for secondary mathematics* supplement of examples, pages 187 to 189, 203 and 215.

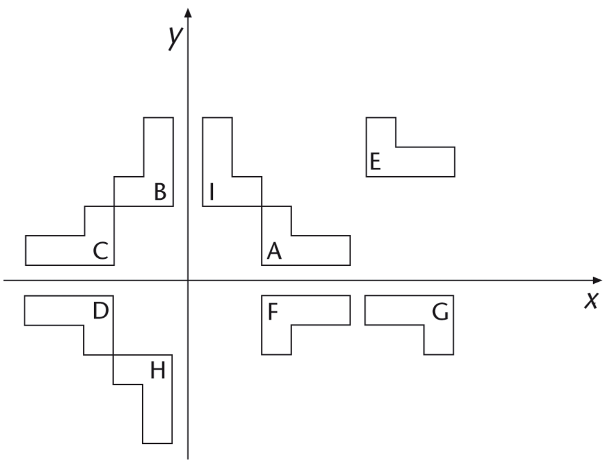
Mathematical activity tiles (MATS) are available from ATM (www.atm.org.uk).

What is the transformation? is a useful activity for identifying multiple transformations, as well as more simple ones. Most transformations on a coordinate grid provoke a mental approach.

- Give the pupils the object and image and ask them to find the transformation.
- Use examples requiring more than one transformation.

Encourage pupils to share their way of understanding the transformations. Some may use analogies that can help others with their mental images, for example, 'The image is like a print of the object if the paper is folded on the line of reflection,' or 'The rotation is like an invisible clock hand, with a shape fixed to the end of it. It can print any number of duplicate shapes as it rotates.'

Even with examples requiring more than one simple transformation, a pupil needs to have a mental picture of both object and image and may use this mental picture as the basis of a practical check, perhaps using tracing paper. See resource 2, *What is the transformation?* on page 36.



Always, sometimes or never true? generates fruitful discussion. Compose a set of statements and ask pupils to classify them as 'always true', 'sometimes true' or 'never true'. For example:

- An octagon has zero, one, two, four or eight lines of symmetry.
- A pentagon has one line of symmetry.
- A quadrilateral has exactly three lines of symmetry.

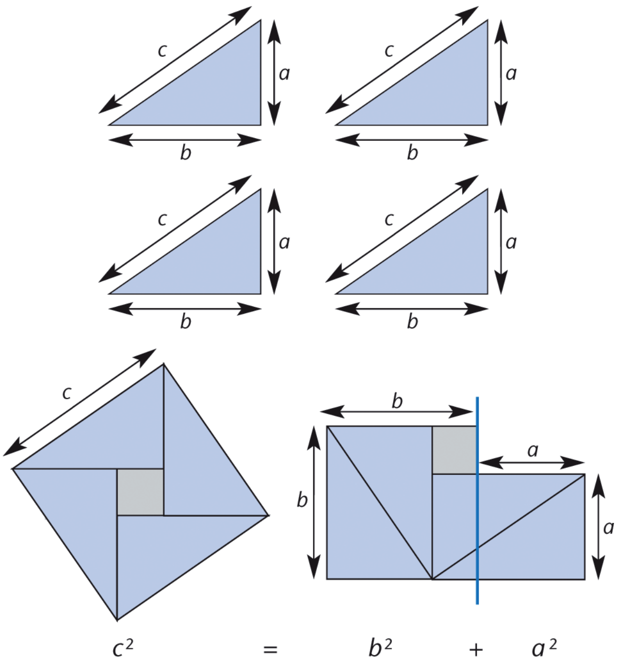
Interventions might include providing some pupils with counter-examples, reminding them, for example, that not all polygons are regular. Pupils must use geometrical argument to justify their conclusions. Follow-up prompts to support pupils' thinking might include:

- How do you know that an octagon *always* has zero, one, two, four or eight lines of symmetry? Why can't it have seven lines of symmetry?
- Is it possible for a pentagon to have just one line of symmetry? What is special about the pentagon in this case?
- Why can a quadrilateral *never* have exactly three lines of symmetry?

Investigating statements such as these can lead pupils to a conjecture that a polygon either has no line of symmetry or a number of lines of symmetry that is a factor of its total number of sides.

Once pupils are familiar with this activity, giving them the opportunity to invent their own statements that are *always true, sometimes true* or *never true* invites them to explore the possibilities open to them of modifying geometrical shapes to change their symmetry properties. The statements they produce can be used in small-group work or in a plenary session to discuss what makes certain statements more challenging than others.

Providing a commentary for a proof of Pythagoras' theorem can develop mental and written reasoning. Pupils familiar with Pythagoras' theorem should understand and be able to explain the dissection proof illustrated below. Put these images in the centre of a large, blank page and ask pupils to annotate the diagrams. They should produce a chain of reasoning to show how the steps build up to produce the final equation.



'Within' shapes is based on the idea of single shapes building up to form larger, composite shapes. Pupils are asked to consider how the 'within' shapes influence properties of the subsequent composite shape. Ask:

- What shapes can you make by reflecting a triangle in one of its sides?
- Does it make a difference if the triangle is equilateral? …isosceles? …right angled?

This is an extension of the activity on page 206 of the *Framework for secondary mathematics* supplement of examples. The task can be started with any shape.

Alternatively, provide the composite shape and ask questions about the 'within' shape, for example:

A hexagon is produced when a polygon is reflected in one of its sides.

- What could the original polygon be? How many solutions are there?
- What properties would the hexagon have?

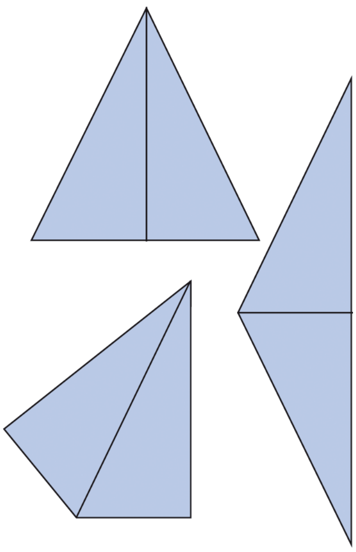
Similar thinking can be prompted by asking about folding the composite shape, for example:

A shape has been folded along a line of symmetry; the result is a right-angled triangle.

- What could the original shape have been?
- How many solutions are there?
- How do you know you have found them all?

Alternatively, provide the composite shape and ask questions about the original shape and the 'within' shape, for example:

- What is the original shape if the result after folding is:
  - a parallelogram
  - an isosceles trapezium
  - an isosceles triangle
  - a right-angled trapezium?



Enlargement and similarity

Enlargements and similarity are applications of ratio and proportion. Making this link plays an important part in helping pupils to see the 'big picture'. Mental methods that pupils develop to solve ratio and proportion problems can be extended to their work in geometry. Equally, work in geometry can provide a context in which pupils are expected to explain and justify the mental approaches that they use. This is crucial if pupils are to extend their understanding of enlargement into trigonometry. The intention is that pupils understand the processes involved and make connections in their own learning rather than struggling to apply a set of 'learned rules' to challenging contextual problems.

Classroom experience suggests that pupils' understanding of enlargements and similarity can be developed through a range of practical and mental activities, for example:

- using centres of enlargement to generate whole-number, fractional and negative enlargements;
- comparing lengths within and between a shape and its enlargement, and distances from the centre of enlargement;
- using ICT and photocopiers to generate enlarged shapes, and discussing scale factors associated with different paper sizes, including the international 'A' series.

Such experiences help pupils to develop their mental imagery as a vital tool in identifying how shapes are enlarged, in recognising their common features and in building strategies that can help them to solve problems involving applications of enlargement and similarity.

Understand and use enlargements and similarity:
recognising and visualising enlargements – positive whole-number scale factors – positive fractional scale factors – negative scale factors
using and interpreting effects of scale factors of enlargements
understanding and applying similarity and congruence
understanding and using Pythagoras' theorem when solving problems in two and three dimensions
beginning to use sine, cosine and tangent in right-angled triangles to solve problems in two dimensions.

The Framework for secondary mathematics supplement of examples, pages 213 to 215, provides contexts in which pupils should develop mental processes in enlargements and similarity.

Activities

Ray visualisers provide an introduction to enlargements by using prepared enlargements shown on lines radiating from the centre of enlargement. This can be done on paper or using technology (ICT).

On **paper**, prepare large display copies or OHT copies of the resource sheets shown on pages 37 to 39. Show resource 3: *Rays of enlargement A*, and ask pupils to imagine a small triangle with a vertex on each ray. Next, ask them to imagine a triangle with sides that are twice as long as in the original, still with a vertex on each ray. Ask:

- How far up the rays does the larger triangle sit?

Show resource 4: *Rays of enlargement B*, and take different responses to these questions:

- Which angles are the same?
- Which lengths are doubled?

Show resource 5: *Rays of enlargement C*, and ask pupils:

- Which of the two triangles is the correct enlargement?

They should explain why.

The challenge can be varied by preparing more resource sheets:

- starting from a larger object shape and enlarging with a scale factor between 0 and 1
- using rays that radiate in both directions to allow for a negative scale factor
- repositioning the centre at a vertex of the object shape, on a side of the object shape or within the object shape
- showing a pair of triangles without rays and asking where the centre could be.

Using **technology** such as dynamic geometry software on an interactive whiteboard (or a computer projection system), create images A, B and C, as on the resource sheets on pages 37–39, and proceed as above.

The software can considerably enhance the task by allowing pupils to think about what happens to the image if the centre of enlargement is moved. Their hypotheses can be tested immediately and subsequently refined.

As pupils drag the centre of enlargement to different locations on the screen, ask the class to speculate about what stays the same and what changes.

By asking appropriate questions, ensure that pupils understand that when the sides of the triangle are doubled in length the distances from the centre of enlargement are also doubled.

Now construct a second image from the original object, using the same centre of enlargement, but this time with a scale factor 3. Ask similar questions for the new shape.

- Which angles are the same?
- Which lengths are trebled?

Then ask:

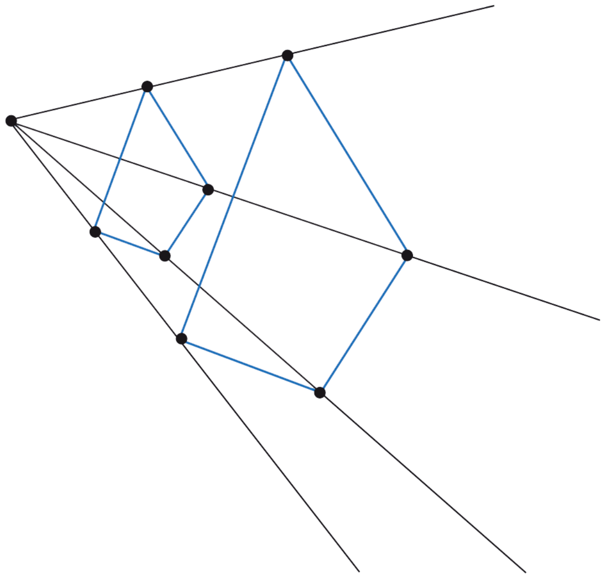
- What would a scale factor of $-\frac{1}{2}$ (or -2) mean? How would you construct the enlargement?

  Is the image in the same position? Why/why not?
- What about scale factors of $\frac{1}{3}$, -3, $\frac{2}{3}$, $-\frac{2}{3}$, -0.1 … ?
- How could you predict their positions?

Take feedback about each of the above scale factors in turn, ensuring that pupils explain and justify their conjectures. Change the scale factor to show the result and discuss the accuracy of their suggestions.

Use this discussion to draw out general principles and applications of enlargements by scale factors greater than 1, between 0 and 1, and less than 0.

The *Framework for secondary mathematics* supplement of examples, pages 213 and 215, provides examples on applications of enlargements.



You can access a dynamic version of this image at **www.chartwellyorke.com/mentalgeometry/enlargements.html**.

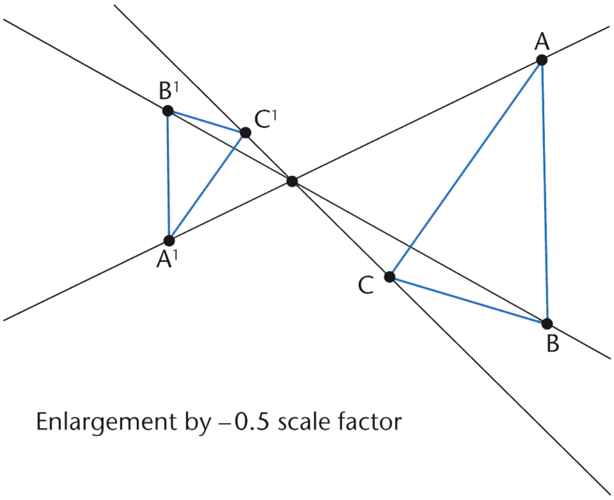
Always, sometimes or never true? can generate interesting discussion. Compile a set of statements and ask pupils to classify them as 'always true', 'sometimes true' or 'never true'. For example:

- If a shape is enlarged by a scale factor 2, then the perimeter of the image is doubled.
- Enlargements produce larger shapes.
- If a shape is enlarged by a scale factor 2, then the area of the image is doubled.

Intervene with pairs who are 'stuck' by asking them to try to visualise or draw one shape that confirms each statement and another that does not. Pupils must use geometrical argument to justify their conclusions. Follow-up prompts might include:

- Is it *necessarily true* that if a shape is enlarged by a scale factor 2, then the perimeter of the image is *always* doubled? Put together a chain of reasoning that will convince another pair.
- Do enlargements *necessarily* produce larger shapes? When does this happen? When does it *not* happen?
- Is it possible for the area of the image to be doubled when a shape is enlarged by a scale factor 2? Explain what happens to the area if the sides are doubled. Under what circumstances does the area double?

When pupils engage in rich tasks their understanding of enlargement and similarity will develop. As misconceptions become apparent, stimulate discussions of the kind described above, perhaps using an 'always, sometimes, never' approach. Further examples on similarity can be found in the *Framework for secondary mathematics* supplement of examples, page 193.



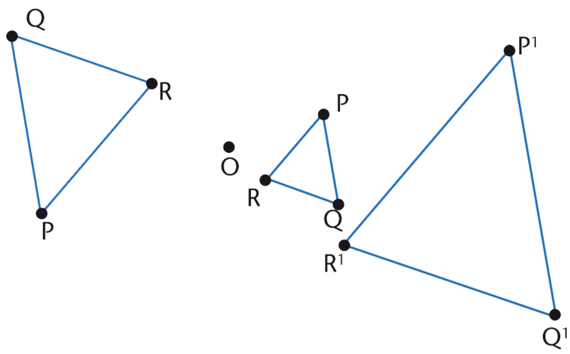
Enlargement by −0.5 scale factor

In **Matching,** pupils make decisions about congruence or, later, similarity of shapes. They are asked to judge, from given information, whether a shape would have the same form, particularly the same angles, no matter how they tried to construct it. Take as an example a parallelogram with sides of 4 cm and 5 cm and one angle of 134°. For most people, this is a mental activity involving checking the orientation and relative positions of angles and sides that correspond to one another. Once pupils have successfully completed the matching process, congruence usually reveals other properties, while similarity tends to lead to work with ratios to establish lengths of sides. Thus a mental matching task could be a precursor to more detailed written checking and justification.

On resource 6: *Enlargement grid,* on page 40, the triangle ABC has been enlarged from different centres of enlargement, using different scale factors, to produce triangles DEF, GHI, JKL, MNO, PQR, STU, VWX, YZA_1 and $B_1C_1D_1$. Ask pupils to locate the centre and scale factor of each enlargement. Tell them to beware because one of the triangles is not an enlargement of the original. Ask pupils to identify which triangle this is and to move one of its vertices so that it is an enlargement of the original.

Once pupils have completed this task, a good plenary activity is to ask pupils to move between the various enlargements. For example, given × 3 and ×–2 enlargements of the same figure, they could be asked:

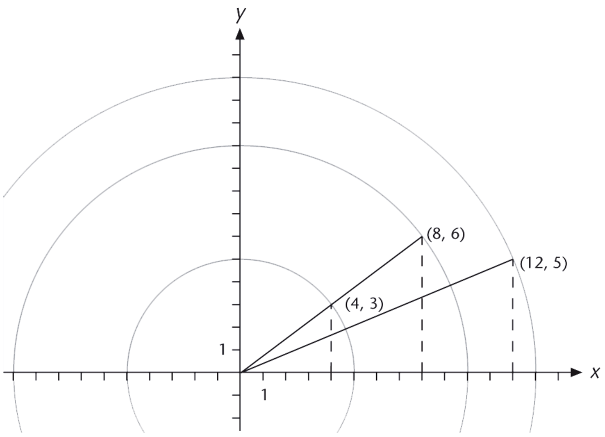
- What scale factor is involved in moving between those two enlargements?
- Where would the centre be?



You can access a dynamic version of this image at **www.chartwellyorke.com/mentalgeometry/enlargements.html**

Providing a commentary requires pupils to work on an illustration of Pythagorean triples. Give each pair of pupils a large sheet of 5 mm squared paper showing concentric circles centred on the origin. Choose radii that are multiples of 5, 13, 17 or 25. Ask pupils to mark, on the circles, points that have integer *(x, y)* coordinates. They should then draw a right-angled triangle with sides connecting the point on the circle to the origin and a point on the axis.

Their task now is to annotate the diagram, which could be pasted in the centre of an A2 sheet of paper. Pairs should use their notes to describe what they notice about the lengths of the sides of each triangle and how the triangles compare to each other.



For example:

- Are the points accurate? The point (13, 11) does not quite lie on the circle of radius 17. How can you use Pythagoras' theorem to check? (Look at the units digits.)
- How are some triangles linked by transformations?

If pupils fail to notice similar triangles, suggest that they look at rays from the origin through some sets of points. For example, for circles with radii 5, 10, 15 and 20 units, the ray through (3, 4) also passes through (6, 8), (9, 12) and (12, 16).

The question, 'How many Pythagorean triples exist?' is a straightforward one, since there are infinitely many enlargements of (3, 4, 5), for example. A more difficult question to answer is, 'If (3, 4, 5), (6, 8, 10), … represent one family of Pythagorean triples, how many different *families* of Pythagorean triples are there?'

Earlier applications of Pythagoras' theorem can be found in the *Framework for secondary mathematics* supplement of examples, page 189.

Photographic enlargement can be used to introduce trigonometry through enlargement. Trigonometry has its roots in similarity. The trigonometric ratios use proportional reasoning to compare any given right-angled triangle with another in which the hypotenuse is of unit length. The idea is developed as an interactive teaching program (ITP) for use on projectors or interactive whiteboards and is available on the *Framework for secondary mathematics*. Understanding the equivalence of trigonometric ratios for similar triangles hinges on understanding that the ratio of sides within similar shapes is maintained under enlargement.

Use the ITP. Explain that:

- This is a set of triangles that are enlargements of one another – it is a set of similar triangles.
- There are 18 values missing from the diagram (*on a poster have these values hidden by small sticky notes*), but they cannot be found by measuring as the triangles are not drawn accurately.
- Of the missing values, six are dimensions (the two shorter sides of the triangle), six are scalings between the three triangles in both directions and six are internal ratios written as scalings from one side to the other and vice versa (these six are the tangents of two acute angles in the triangles).

Set the class this challenge.

I will give you some of the values. You have to ask for the minimum you need to work out all the other values. What values do you want me to give? (Which flaps will you choose to lift?)

Use the task to focus on the fact that the internal ratios are the same for all the triangles regardless of the scalings between them.

Discussion should also include the fact that knowing a side and an internal ratio means that you do not need to reveal the other side because it can be calculated.

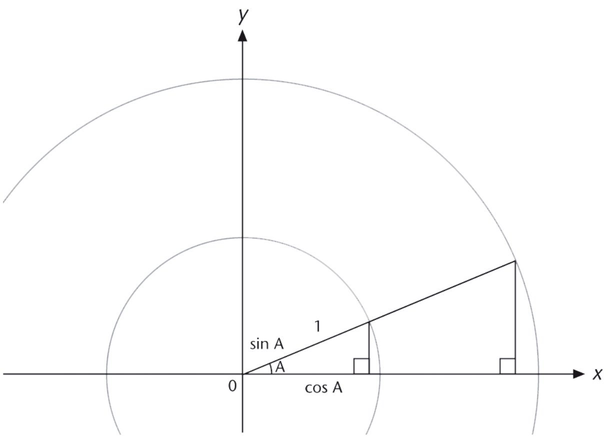
The *Framework for secondary mathematics* supplement of examples, pages 191 and 193, includes examples on congruence and similarity. These are supported by interactive software available from the website of the Mathematical Association (**www.m-a.org.uk**).

Triangles on a unit circle can be used as a follow-up to photographic enlargement and as a precursor to work on the rotating unit arm to generate trigonometric curves. Effectively, the pupils are asked to annotate nested similar triangles based on the connections they have formed from working with the photographic enlargement task.

Give each pair of pupils a large sheet of 5 mm squared paper showing concentric circles centred on the origin. On these diagrams pupils should draw some sets of similar right-angled triangles with a vertex at the origin, as shown below. The diagrams could be pasted in the centre of an A2 sheet of paper for pairs to add annotations describing what they know about:

- how the triangles compare to each other;
- the ratio of the lengths of pairs of sides within a set of triangles.

The concluding discussion should establish that ratios in the triangles on the unit circle can be generalised, through enlargement, to every right-angled triangle.



You can access a dynamic version of this image at **www.chartwellyorke.com/mentalgeometry/unitcircle.html**.

Sorting right-angled triangles is a matching activity that supports pupils' reasoning about, and facility with, trigonometry. It is an end-of-unit or revision activity, rather than an introduction to the topic. The pupils should not work with pencil and paper or a calculator. The aim of the task is to increase the facility with which pupils connect the dimensions shown on the triangle with the trigonometric ratio.

In the activity, pupils are provided with cut-out copies of cards produced from resource 8: *Right-angled triangles for sorting,* on page 42, and resource 9: *Trigonometric ratios for matching,* on page 43. Working in pairs or small groups, they sort the triangles according to whether they are similar and match the trigonometric ratios for the angle marked in each triangle. In the triangles, some of the ratios repeat, so the numbers of cards in the two sets are not equal. Please note the additional challenge involved in sorting some cards, which involves simplification of surds, such as $\sqrt{45} = 3\sqrt{5}$.

To increase the challenge further, pupils could be asked to reduce the number of matched sets to a minimum by merging equivalent sets. This involves mental use of Pythagoras' theorem, for example, the set for $\cos x = \frac{1}{\sqrt{2}}$ is equivalent to the set for $\tan x = 1$.

The *Framework for secondary mathematics* supplement of examples, pages 191 and 193, contains examples on congruence and similarity. These are supported by interactive software available from the website of the Mathematical Association (**www.m-a.org.uk**).

Constructions and loci

Traditionally, when solving geometric problems, pupils are expected to analyse given diagrams, linking the properties to known facts, and to derive or prove new facts. The development of dynamic geometry software has transformed this process. Through this medium it is possible to construct diagrams accurately, then repeatedly change their appearance without affecting key properties. The analysis of such images, identifying what has changed and what has stayed the same, provides pupils with fresh insights into ways of solving geometric problems. Use of dynamic geometry software transfers the focus onto the mental processes underpinning the mathematics, rather than pupils' technical competence in drawing. This latter skill is still important in mathematics, but it is different from the ability to reason in geometry. In short, the rapid construction of shapes and diagrams, and their subsequent transformation, help pupils to recognise geometric features.

The work on loci is embedded within mental mathematics because the imagery is vital in helping pupils to predict and, later, to check solutions. The geometrical constraints placed upon loci produced on paper require a good understanding of the fundamentals of geometrical reasoning and of constructions. Visualisations, 'people geometry' activities and the use of Logo all support pupils in developing their understanding of loci, leading to the solution of practical and mathematical problems. Dynamic geometry software also provides an important medium for exploring pupils' conjectures about loci, for discussing dependent and independent points and constraints on movement.

For all learners, spending time learning to recognise properties of shapes, through construction and testing hypotheses about the paths of moving points, is likely to sustain their confidence and increase their competence with more complex geometrical problems.

Recognise, construct and explain constructions and loci by:
recognising and visualising constructions of triangles and nets of 3-D shapes
devising instructions for a computer to generate and transform shapes and paths
determining the locus of an object moving according to a rule
understanding and explaining links between loci and constructions and the properties of shapes
investigating and visualising solutions to problems involving loci and simple constructions

The *Framework for secondary mathematics* supplement of examples, provides contexts in which pupils should develop their mental imagery in the areas of:

- constructions and loci; (pages 220 to 227)
- associated properties of triangles, quadrilaterals and polygons; (pages 184 to 189)
- properties of circles. (pages 195 and 197)

Activities

Construction visualisation 1, instructions to pupils:

Close your eyes. Imagine a circle with centre A and a second circle of different radius with centre B. Slide one circle partly over the other. Imagine a line drawn joining the centres A and B. Now imagine a line segment joining the two points of intersection of the circles. Look at where these two lines cross. What do you notice?

Open your eyes and discuss the image with a partner.

- How would you explain why the lines cross at right angles?
- Can you prove this?
- What given fact are you using in this proof?
- What would change if the two circles were identical?
- Why?

Construction visualisation 2, instructions to pupils:

Close your eyes. Imagine a line and a point not on the line. Imagine a family of circles, all passing through the point and just touching the line. What is the position of the smallest of these circles? What about the largest? As the circles shrink and grow in size, can you see a path formed by the centres of the circles?

Open your eyes and discuss the image with a partner.

Pupils may not see that the path of the centres is a parabola. This could be confirmed by further exploration with counters. Once a few counters have been placed, ask pupils to imagine what the path will look like. Let them test conjectures and any stray points by measuring. This is also a suitable **People geometry** task.

Once pupils have constructed their mental imagery, the use of **dynamic geometry software** to construct the images can support pupils as they discuss and test their conjectures. The construction of the second visualisation requires further geometrical knowledge, particularly that the perpendicular bisector of any chord of a circle passes through its centre.

People geometry can make it easier for pupils to think about loci by working on them practically. Ask pupils to explore problems by moving themselves or placing counters according to a set of instructions. Using string to check that the conditions of the original problem are being fulfilled is preferable to the distraction of numerical measurement.

The string focuses attention on the equality of lengths or ratio of lengths rather than the numerical value and thus teases out the generality of the solution. Even without precise measurement and exact calculation, pupils can deduce properties that link with familiar shapes.

Further ideas to support this activity are available on resource 10: *People geometry loci cards,* on page 44**.** See also the *Framework for secondary mathematics* supplement of examples, pages 15 and 225 to 227.

Thinking around a rhombus encourages pupils to think carefully and creatively about constructing a rhombus, Ask:

- Using dynamic geometry software (or pencil, ruler, set square and compasses), in how many different ways can you construct a rhombus?

There are many ways. Even without using dynamic geometry, pupils should be able to think of at least four. Ask them to compare the geometric properties they have assumed in each method that they find. Use of dynamic geometry software significantly increases the number of possibilities, as pupils can use reflections and rotations. Pulling together a 'class set' of methods can make a useful plenary, emphasising the ways in which geometric properties link together to define the shape.

The activity can easily be extended to other quadrilaterals. This task encourages pupils to consider a wide range of properties of geometrical shapes when they are solving problems.

Loci visualisation 1, instructions to pupils:

Close your eyes. Imagine two 10p coins, placed flat on the table and edge to edge. Now imagine rolling one around the other. Describe the path of the centre of the moving coin. How many times has this coin rotated by the time it returns to its starting point?

Open your eyes and discuss the image with a partner.

- Did you both see the same path?
- Do you agree about the number of rotations?

Loci visualisation 2, instructions to pupils:

Close your eyes. Imagine a 10p and a 5p coin placed flat on the table and edge to edge. Roll the 5p coin around the 10p coin. What path does the centre of the 5p coin take as it rolls around the 10p coin? Imagine a point on the edge or circumference of the 5p coin. What path does the point trace as it moves around the 10p coin? Now imagine a point on the horizontal surface of the 5p coin but not at the centre. Describe its path as the coin rolls around the 10p coin.

Open your eyes and discuss the image with a partner.

- Did you both have the same image?
- Do you agree about the changes that result from looking at different points?

Loci visualisation 3, instructions to pupils:

Close your eyes. Imagine a wheel moving on an axle, in the usual way, as a cart moves along horizontal *ground. Imagine a point on the edge or rim of the wheel. Describe the path of this point as the wheel moves along the ground. What happens for a larger wheel? For a smaller wheel?*

Open your eyes and discuss the image with a partner.

- Did you both have the same image?

Loci visualisation 4, instructions to pupils:

Close your eyes. Imagine a 10p coin and a small triangular tile, placed flat on the table and edge to edge. Now imagine rolling the coin around the tile. Describe the path of the centre of the moving coin. Describe the path of a point on the edge or circumference of the coin.

Open your eyes and discuss the image with a partner.

- Did you both have the same image for each position of the moving point?

Loci visualisation 5, instructions to pupils:

Close your eyes. Imagine an equilateral triangle 'rolled' along a line segment. Describe the path of the centre point of the triangle. Now imagine a point on one of the sides of the triangle and describe its path. Imagine a point at one of the vertices of the triangle as it moves along the line and describe the path. Now imagine the triangle reaching the end of the line segment and rolling along the under side. Describe the new locus for each case described above.

Open your eyes and discuss the image with a partner.

- Did you both have the same image?
- Which part was difficult to visualise? Why?

See also QCDA booklet, *Developing reasoning through algebra and geometry*, page 26, 'The midpoint of a sliding ladder'. (ISBN 1 85838 550 4)

Logo polygons

Ask pupils to write a Logo procedure to draw an equilateral triangle of side 60. Ask how they would adapt this procedure to construct other regular polygons:

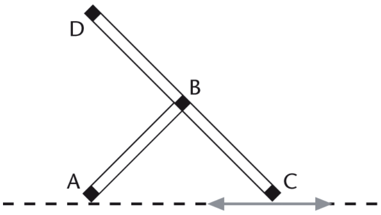
- a square
- a hexagon
- a pentagon
- an octagon.

Ask them to describe how to construct a regular polygon with any given number of sides, using Logo.

Ask pupils how they could adapt the procedure for drawing the equilateral triangle to produce a six-pointed star.

There are many ways of doing this. One example is: repeat 3 [fd 60 lt 60 repeat 2 [fd 60 rt 120] bk 60 lt 60]

Loci in the 'real world' are initially, at least, tackled mentally. Objects or mechanisms are there to be seen and people look at how they operate, and only later might they think about the design of these mechanisms and why they work in that way. Such an example is the shower cubicle shown in the *Framework for secondary mathematics* supplement of examples, page 227.



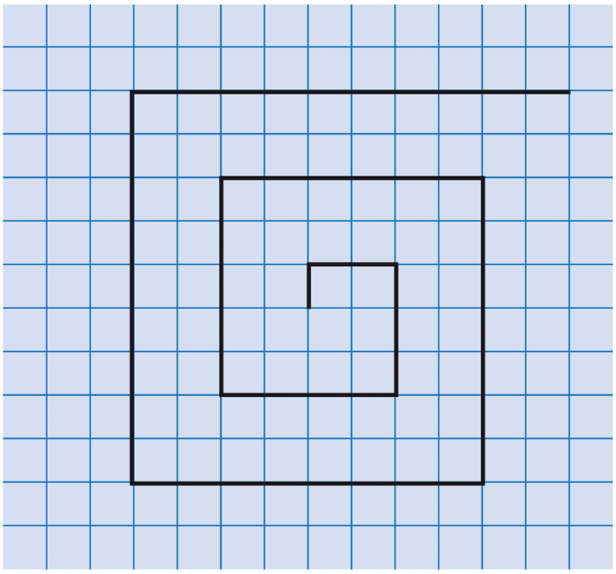
Ask pupils to think about 'The Spider', a small **theme-park** ride with eight 'legs' rotating clockwise from a central hub. On the end of each leg is another hub with four smaller arms, which also rotate clockwise. On the end of each arm is a chair. Ask:

- Imagine you are sitting in one of these chairs. Describe your locus, once the Spider starts in motion.
- Assuming the spectators all stand still, would you see the same faces each time you reach the edges of the ride?

Other examples of loci in the real world can provide useful links from geometry to algebra. For example, using photographic images, the path taken by water from a spout *or* the locus of a basketball from a free throw to the basket can be seen to take the form of parabolae with equations that can be found by matching quadratic graphs to the photographs.

Photographic images to support this work can be found in the Maths Gallery on the ATM CD-ROM, *Integrating ICT into the mathematics classroom* (**www.atm.org.uk**), distributed to all schools in November 2005.

In **sport**, some loci are produced through the use of highly sophisticated mathematical models. One such mathematical model is used by television presenters to verify 'lbw' decisions in cricket. The projected path of a cricket ball is animated on screen to see where it would travel had it not hit the batsman's pad. The variables in the mathematical model may include factors such as the firmness of the ground, the type of bowling, the number of times the ball had been bowled. These will be constantly updated, based on the behaviour of the ball on the pitch. This is beyond level 8 mathematics but serves to illustrate the potential of loci as a feature of mathematical modelling.



In Logo, spirals can be created by using a variable, such as **:side**, which can be increased incrementally.

The above spiral can be created from a procedure such as:

```
to spiral :side
make side 20 repeat 10 [fd :side rt 90 make side :side + 20]
end
```

The instruction, spiral 20, will produce the required shape.

This spiral is based upon a square. Encourage pupils to investigate other spirals, such as those based upon a pentagon or hexagon.

Working in three dimensions

Pupils will improve their skills in geometrical reasoning in three dimensions by handling solid shapes, making them by constructing nets and exploring how they can fit them together. These experiences are more effective when combined with the development of appropriate vocabulary and opportunities to practise and develop their visualisation skills. These skills are crucial for solving problems in three dimensions, such as those involving Pythagoras' theorem and trigonometric ratios.

Opportunities for visualisation can include recognising symmetries, working with plans and elevations, and problems involving plane sections of solids. Progression in tasks involving physical construction of 3-D shapes could start from the use of linking cubes in Year 7, progress to the development of mental imagery and build to the construction of more complex shapes in subsequent years. Opportunities to solve 3-D problems that involve pieces that are not cubes, for example, puzzles such as the Soma cube or those involving 3-D arrangements of pentominoes, are also important. If pupils have no experience of handing manipulative materials to build compound shapes, they are unlikely to be able to understand the properties of their component parts. Pupils also need to understand that on paper they can only draw 3-D solids as 2-D diagrams. Therefore there are certain accepted conventional representations of standard solids; for example, with the appropriate heading, a 'hexagonal' diagram may represent a cube. In non-standard cases, however, it may not be clear what is being represented unless the object can be observed from a different viewpoint. For example, a drawing may appear from one viewpoint to be a 2 by 2 by 2 cube but another view could reveal any number of cubes from 7 to 11. Try it!

Computer packages for 3-D geometry produce different views, providing a way for pupils to investigate such images dynamically. They enable the pupils themselves to generate images of shapes in 3-D and to explore their properties. No matter how the image is presented the pupils still have to learn to understand a 2-D image of a 3-D object. Many pupils will need time and experience in comparing a physical model with the computer image.

Properties of 3-D objects are listed in the *Framework for secondary mathematics* both in section 4, the supplement of examples and section 5, the vocabulary checklist. Remind pupils about simpler 3-D objects such as the cone and octahedron.

Recognise and use common 2-D representations of 3-D objects and deduce some of their properties by:
using 2-D representations and oral descriptions to visualise 3-D shapes and deduce some of their properties
knowing and using geometric properties of cuboids and shapes made from cuboids
beginning to use plans and elevations
designing nets of solids made from cubes and rectangles
exploring possible 2-D representations of 3-D objects
analysing 3-D shapes through 2-D projections and cross-sections, including plans and elevations

The *Framework for secondary mathematics* supplement of examples, pages 198 to 201 and 207, provides contexts in which pupils should develop mental processes in relation to the geometry of 3-D objects.

Activities

Quadrilateral viewer is an activity concerned with two-dimensional shapes but the context and the mathematics are both three-dimensional. Many pupils find difficulty seeing two-dimensional shapes within a three-dimensional model, and this activity provides a simple introduction to visualising sections through solids, an important skill in mensuration.

Each pupil will need a piece of stiff card with a small square (side about 2 cm) cut out from somewhere near the middle, so they can see through. Ask each pupil to hold up the card and look at the square. Explore with them how they can make their square look bigger (by moving it closer to their eyes) and smaller (further away). Then ask:

- Without bending the card, how can you make the 'viewer' look like a rectangle? (tilting the card away from them)
- How can you make it thinner or fatter?
- Can you make:
  - a 'tall' rectangle
  - a rhombus
  - a parallelogram?
- Can you make any other quadrilaterals?

If a larger piece of card with a square cut out is available, pupils can demonstrate solutions to the whole class by holding it a particular way in front of a light source such as a projector.

The activity provides a useful link to work on sections through a shape such as a square-based pyramid. Using a physical model, pupils could suggest how such a pyramid would need to be cut to create different sections. With the 3-D model it is possible to take sections that are not quadrilaterals. Pupils should be encouraged to discuss the possibilities.

Differences and similarities is a task well suited to the comparison of properties of two or more shapes. Carefully choose two or more solid shapes. Ask pupils to discuss and note differences and similarities, making use of key words such as *faces*, *vertices*, *edges*, *curved*, *planes of symmetry* and *rotational symmetry*.

Vary the degree of challenge by adjusting the supporting resources. For example, pupils could:

- work in pairs and each handle shapes from a set of polyhedra
- be shown large display models that they are not allowed to handle
- be given 2-D representations of 3-D shapes
- work from just the name and a mental image of the solid shapes.

Sharing the image involves one pupil interrogating another in order to identify a mystery 3-D shape. The mystery shape is visible only to the answering pupil, as a sketch or object or on the screen of a computer running dynamic geometry software. The interrogating pupil must identify the shape by asking questions that require only 'yes/no' answers, using key vocabulary such as *faces*, *vertices*, *edges*, *planes of symmetry*. The aim is to identify shapes by asking as few questions as possible. This task helps pupils to communicate accurately and concisely, to develop their ability to transmit and receive information, to enhance their powers of mental imagery and to develop mathematical vocabulary.

Several short visualisation activities are suggested in the *Framework for secondary mathematics* supplement of examples, pages 198 to 199.

Exploring duals is an activity that many pupils find fascinating. Ask pupils to work together, in pairs, to discuss and visualise the dual of a shape as follows.

Hold up a skeleton of a cube and ask:

- How many faces does this shape have?
- How many vertices?
- How many edges?

Ask them to imagine locating the centre of each face, then to join each centre to the centres of the adjacent faces, by straight lines. Ask:

- The lines joining these centres form the skeleton of another 3-D shape. What shape will it be?
- This shape can be called the **dual** of the first shape. When is the dual a similar shape to the original?

Pupils should find that a shape is self-dual if the number of faces is the same as the number of vertices.

Further examples are suggested in the *Framework for secondary mathematics* supplement of examples, page 201.

Visualising changes sets pupils the task of explaining the effects of manipulating standard polyhedra. For example, set pupils these tasks.

- Shear a cuboid and describe the faces, paying attention to whether all images are similar.
- Truncate a cube, by slicing off a single vertex, and describe the shape of the new face. Consider the shapes of other faces that have emerged and what happens as the cut gets bigger.
- Truncate a cube by slicing off all eight vertices equally.

In each case pupils should be encouraged to produce a chain of reasoning to explain how they know what is happening to the shape.

Linking solids and nets provides pupils with the opportunity to link mental images in two and three dimensions. Pupils need practice in working both ways:

- using a model of the solid shape to help them visualise possible nets;
- using the 2-D net to help them visualise the solid shape.

Ultimately, these mental images should become interlinked. Commonly, low-level tasks based on nets involve identifying which arrangements of six squares represent possible nets for a cube. More advanced work should achieve a similar level of confidence with sketching and identifying possible nets for other 3-D shapes such as regular tetrahedra, octahedra, icosahedra and square-based pyramids.

The mental activity of linking solids and nets will inform calculations for accurate construction of 3-D shapes from card and, in mensuration, clarity about the location of angles and distances within 3-D objects.

See the *Framework for secondary mathematics* supplement of examples, pages 14 and 15.

Always, sometimes or never true? generates useful discussion. Ask pupils to imagine using four interlinked cubes to make a shape. Compose a set of statements that pupils classify as 'always true', 'sometimes true' or 'never true'. For example:

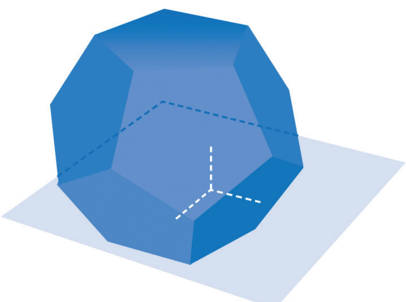
- The shape formed by the four cubes has at least one plane of symmetry.
- The shape formed by the four cubes has two planes of symmetry.
- The shape formed by the four cubes has four planes of symmetry.

They must use geometrical argument to justify their conclusions. Follow-up prompts might include:

- What are the circumstances that ensure that there is at least one plane of symmetry? Describe or show me a shape that has no plane of symmetry at all.
- What would have to happen for the solid to have two planes of symmetry? Is it possible to arrange four cubes in that way?
- Which would be the most likely arrangement of four cubes that would have four planes of symmetry? How many planes of symmetry are then generated? Why doesn't it work?

Further examples are available in the *Framework for secondary mathematics* supplement of examples, page 207.

Wise words requires pupils to describe succinctly what they see. They will need a set of eight cards, each showing a different polyhedron. One pair of pupils choose one image and make two statements about it. Each statement must include only one key word *(faces, vertices, edges, regular, semi-regular)* but must not include the name of the polyhedron. The other pair tries, by asking as few 'yes/no' questions as possible, to identify the shape.



You can access a dynamic version of this image at **www.chartwellyorke.com/mentalgeometry/dodecahedron.html**.

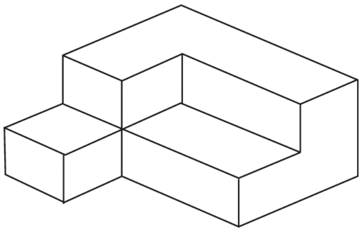
Can you find it? is an activity in shape recognition. Pupils select from a set of polyhedra according to criteria such as:

- a solid with exactly one square face
- two different solids with the same number of faces
- a solid made up of only one type of polygon
- a solid made up of only two types of polygon.

Vary the degree of challenge by adjusting the supporting resources. For example, pupils could:

- work in pairs and each handle shapes from a set of polyhedra
- be shown large display models that they are not allowed to handle
- be given 2-D representations of the 3-D shapes
- work from just the name and a mental image of the solid shapes.

Hidden cubes assists pupils in thinking in three dimensions. Choose a shape such as the one shown below.



Ask:

- How could this solid be made from just nine interlinking cubes?
- What is the maximum number of cubes that could be used to produce the same diagram?
- From which viewpoint(s) would you sketch if you wanted to show either of the above facts?

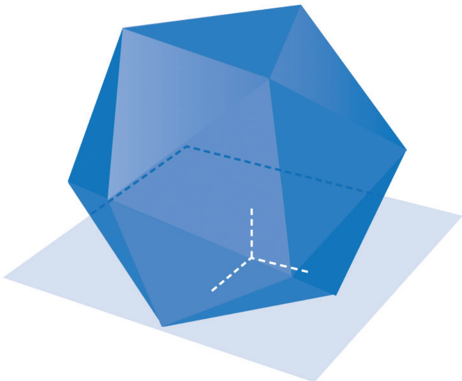
See the *Framework for secondary mathematics* supplement of examples, page 201.

Just eight cubes? links to the previous example. Ask pupils to build a shape in which the number of cubes in the figure represented by the diagram is exactly eight but appears to be greater or less than eight.

Alternatively, ask pupils to create two solids that appear to consist of eight cubes but do not. One solid should use as few cubes as possible and the other should use as many as possible.

For each of the above, pupils should sketch their designs from two views, one that hides any extra cubes or 'holes' and the other that reveals the actual arrangement.

Working on platonic (and other) solids gives pupils necessary experience in using sets of regular polygonal mats with sides of the same length, such as ATM mathematical activity tiles. They could also use 3-D dynamic geometry software. The activity arises out of the *Transformations and tessellations* activity on page 13 of this booklet, with pupils looking at combinations of tiles that do not tessellate, but leave a space.



You can access a dynamic version of this image at **www.chartwellyorke.com/mentalgeometry/icosahedron.html**.

Pupils should speculate about what happens when the edges bordering the space are brought together. They will find that the shape formed comes out of the plane and becomes three-dimensional. Further systematic investigation should result in conclusions such as:

- Four squares meeting at a point leave no gaps, have no overlaps and thus tessellate. Three squares meeting at a point do not tessellate in 2-D but, if pushed together, would form the vertex of a cube in 3-D. (Note that it requires at least three polygons to create a vertex of any polyhedron.)
- Working mentally or physically with the tiles, six equilateral triangles can be shown to tessellate. If fewer than six are used, regular polyhedra can be made. (Five triangles at a point produce an icosahedron; four at a point, an octahedron; three at a point, a tetrahedron.)
- Working with other single regular polygons will show that not all have the potential to form a regular polyhedron. (Five regular pentagons at a point produce part of a dodecahedron.)
- Pupils should draw their ideas to a conclusion by suggesting how many *platonic solids* there are. They should produce a chain of reasoning to justify the number they suggest.

Extending this activity to different combinations of polygons assembled in a regularly repeating manner at each vertex produces a wider range of polyhedra.

Shapes such as the cuboctahedron, with two regular octagons and an equilateral triangle at each vertex, can also be formed by slicing symmetrically across each vertex of a cube. Ask pupils:

- Which 2-D shapes would make the faces of the new 3-D solid?
- What would the new solid look like?

Proceed in a similar fashion for a truncated octahedron and a truncated tetrahedron.

Such visualisation activities, even when linked to the physical construction of a 3-D object, enable pupils to reason and make connections in 3-D, an important skill for life as well as within mathematics.

Mathematical activity tiles (MATS) are available from ATM at **www.atm.org.uk**.

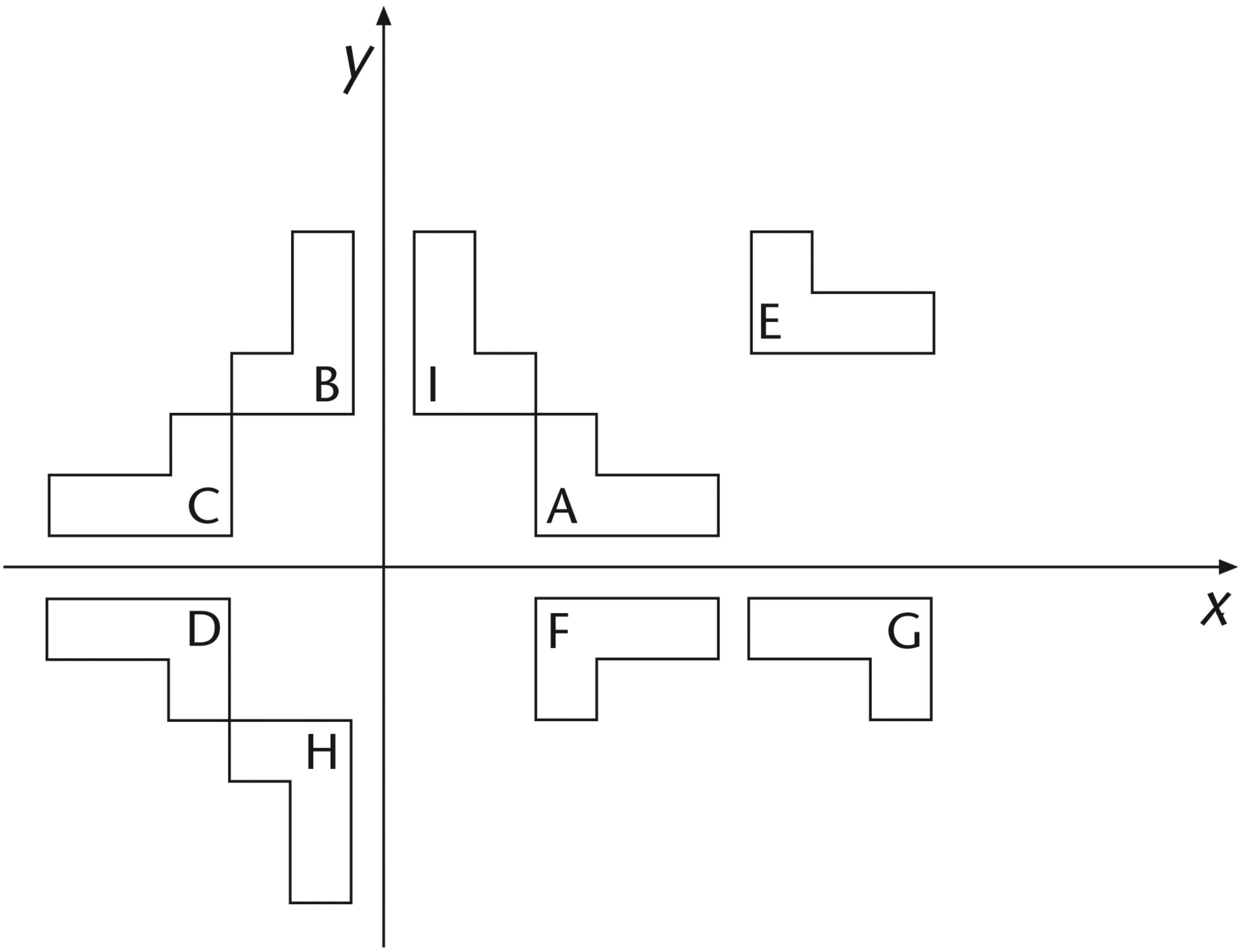
Resources

Resource 1: Sort the properties

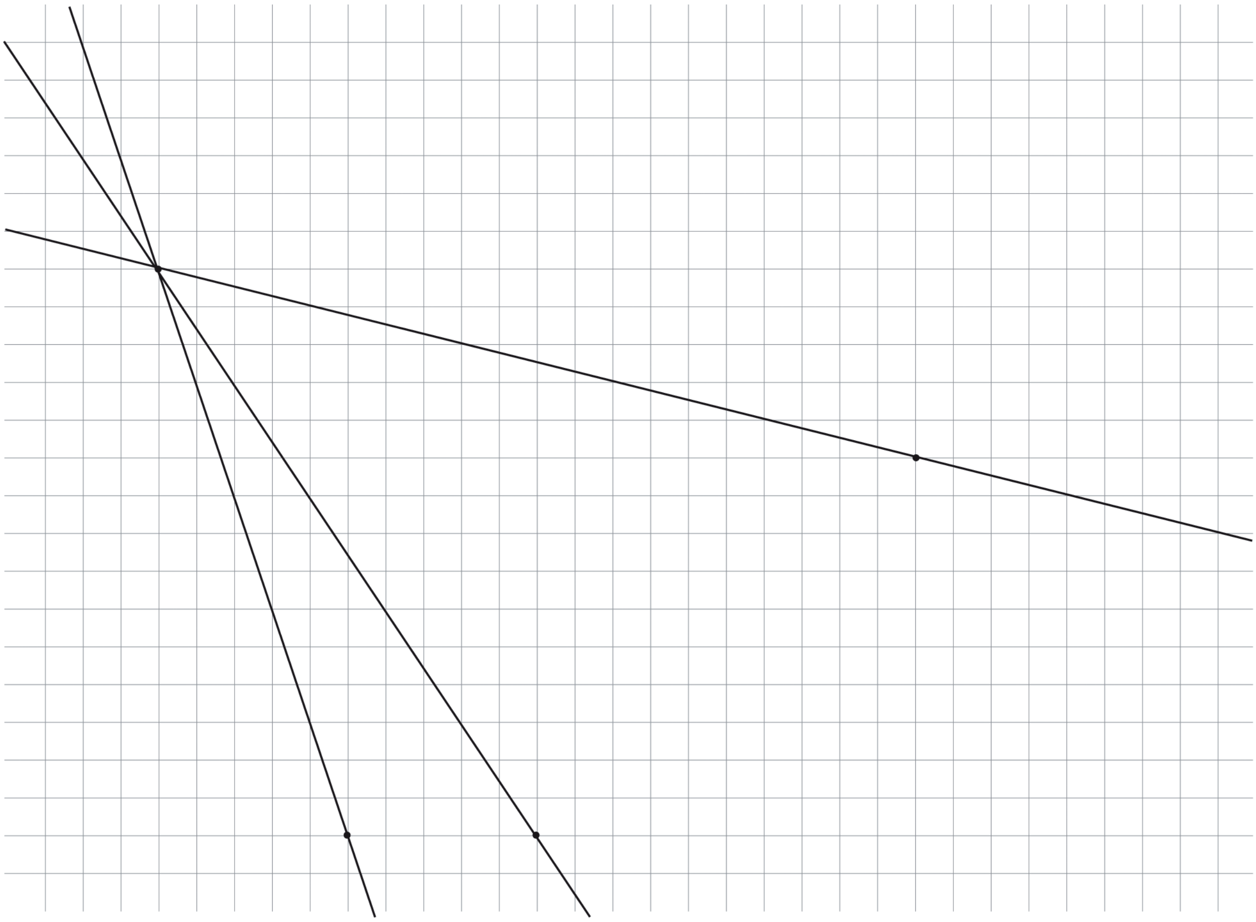
A quadrilateral has	A quadrilateral has	A quadrilateral has
four sides	four angles	two diagonals
A kite has	A kite has	A kite has
one line of symmetry	two pairs of adjacent equal sides	diagonals that cut at right angles
A trapezium has	A trapezium has	A trapezium has
one pair of parallel sides	sometimes, one line of symmetry	sometimes, equal diagonals
A parallelogram has	A parallelogram has	A parallelogram has

two pairs of parallel sides	two pairs of equal sides	diagonals that bisect each other
A rhombus has	A rhombus has	A rhombus has
four equal sides	diagonals that cut at right angles	two lines of symmetry
A rectangle has	A rectangle has	A rectangle has
equal diagonals	four right angles	two lines of symmetry
A square has	A square has	A square has
four equal sides	four lines of symmetry	four equal angles

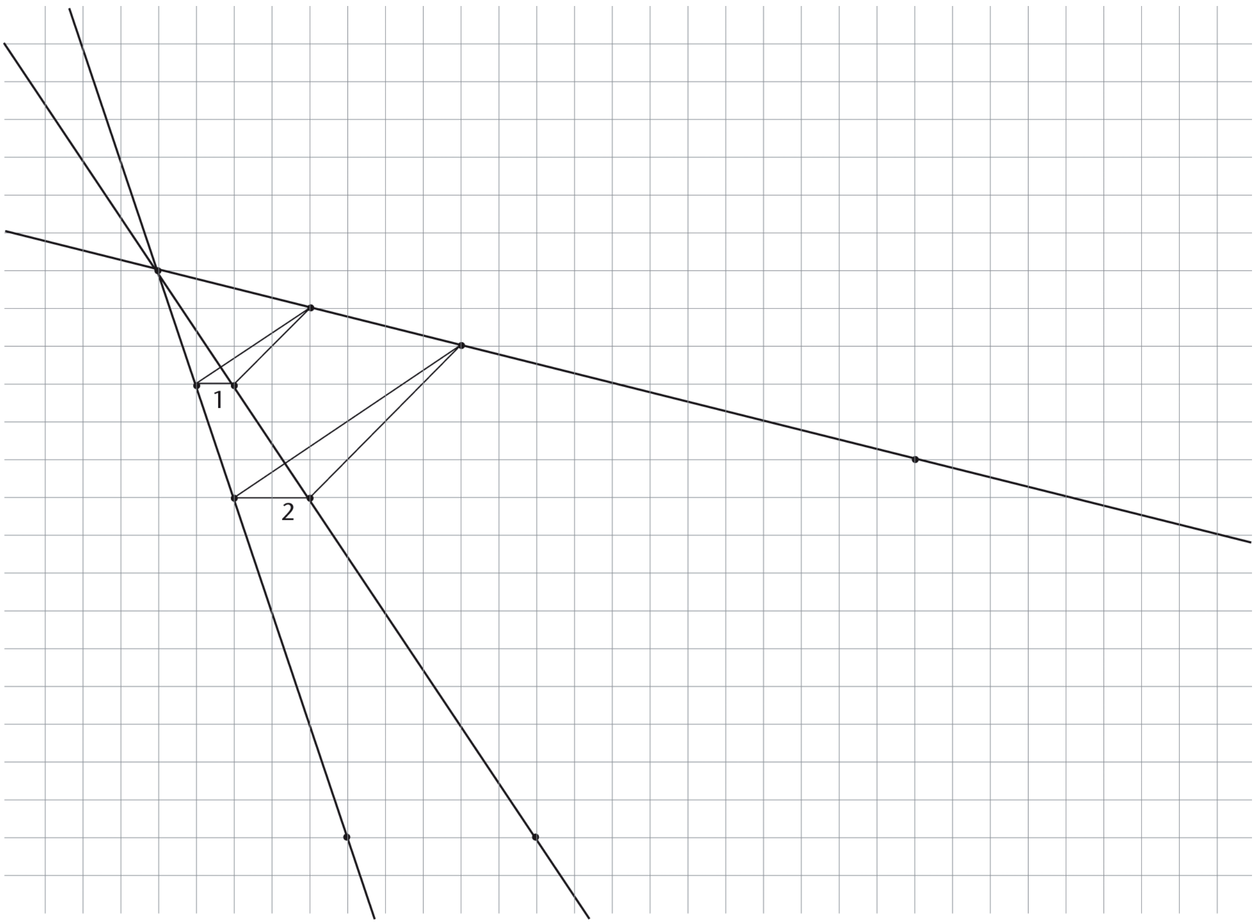
Resource 2: What is the transformation?



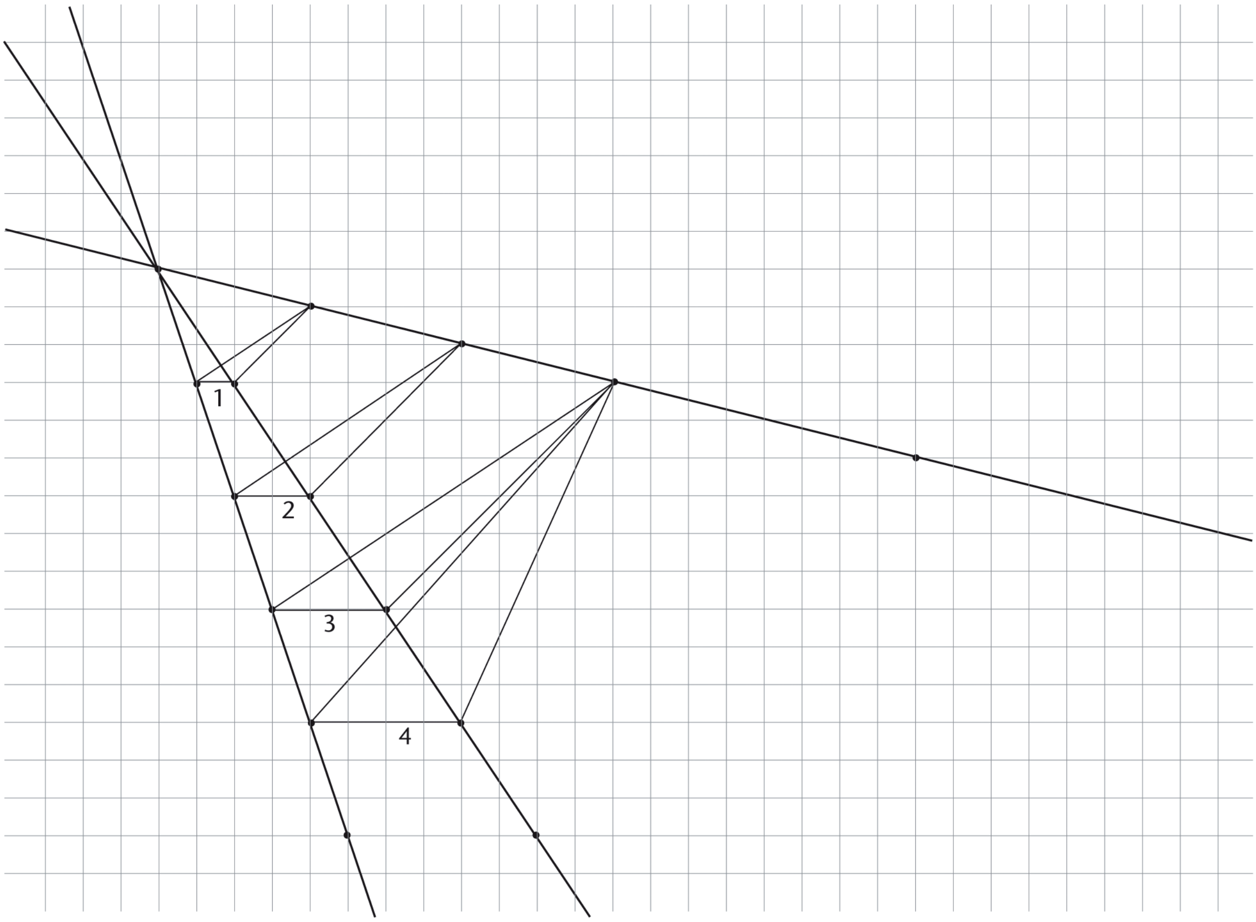
Resource 3: Rays of enlargement A



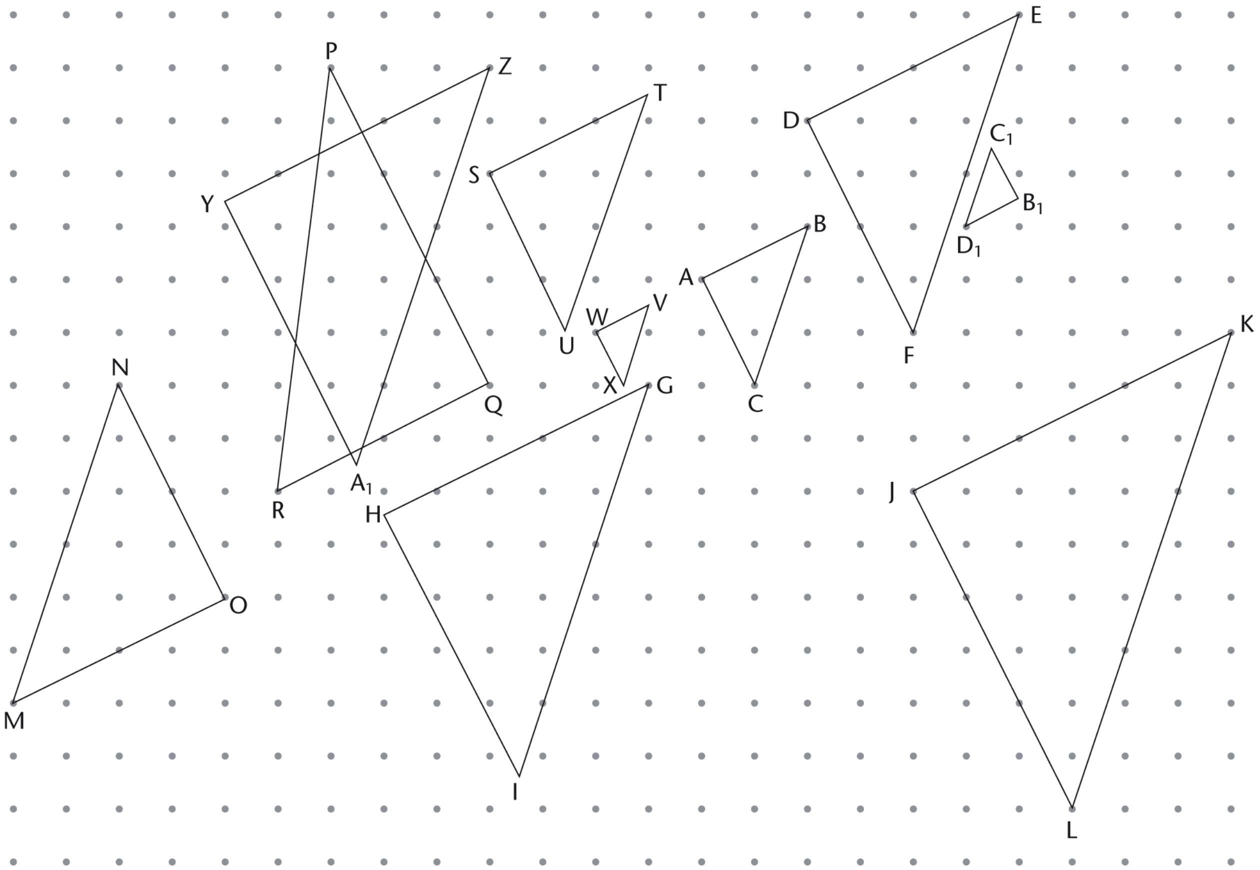
Resource 4: Rays of enlargement B



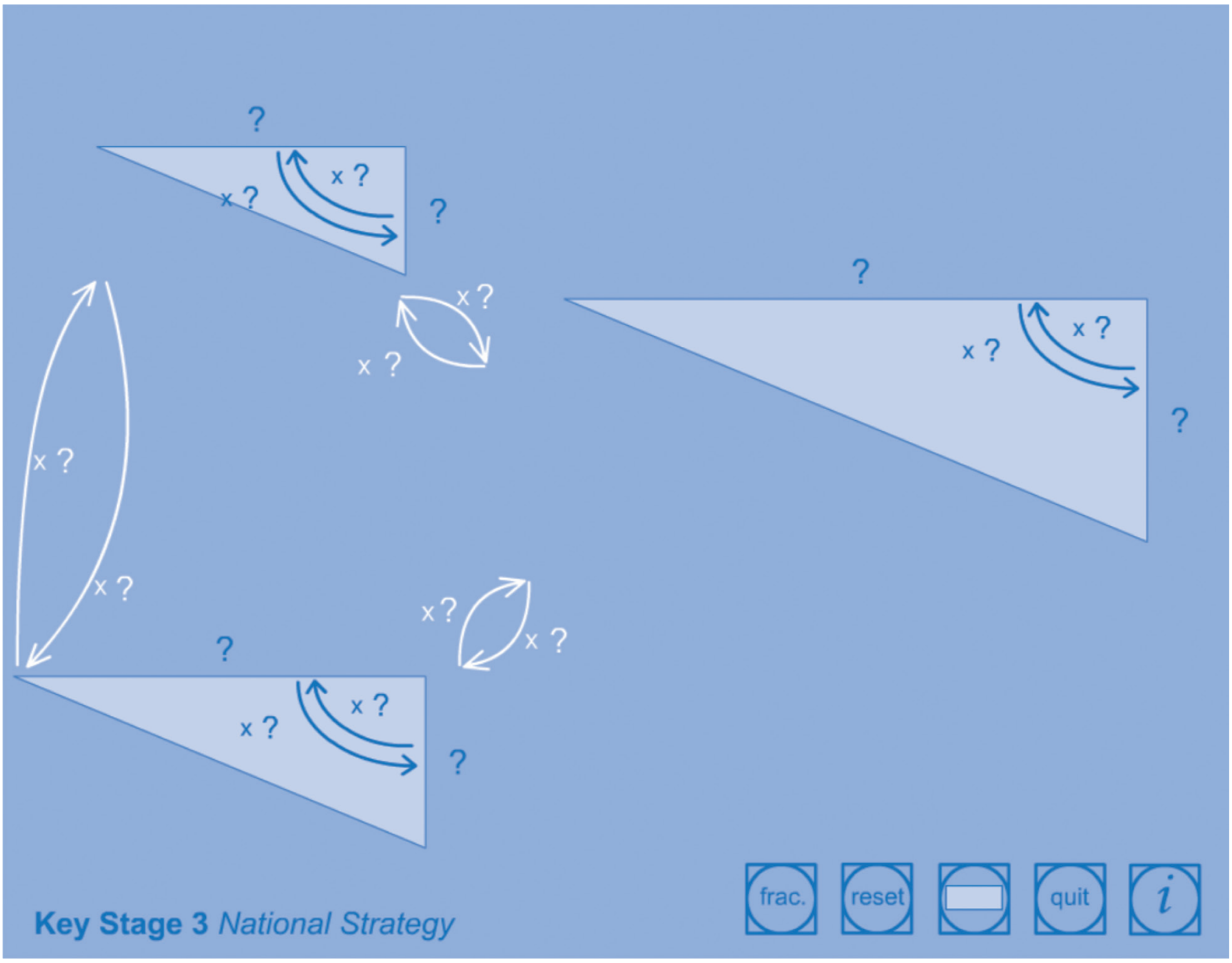
Resource 5: Rays of enlargement C



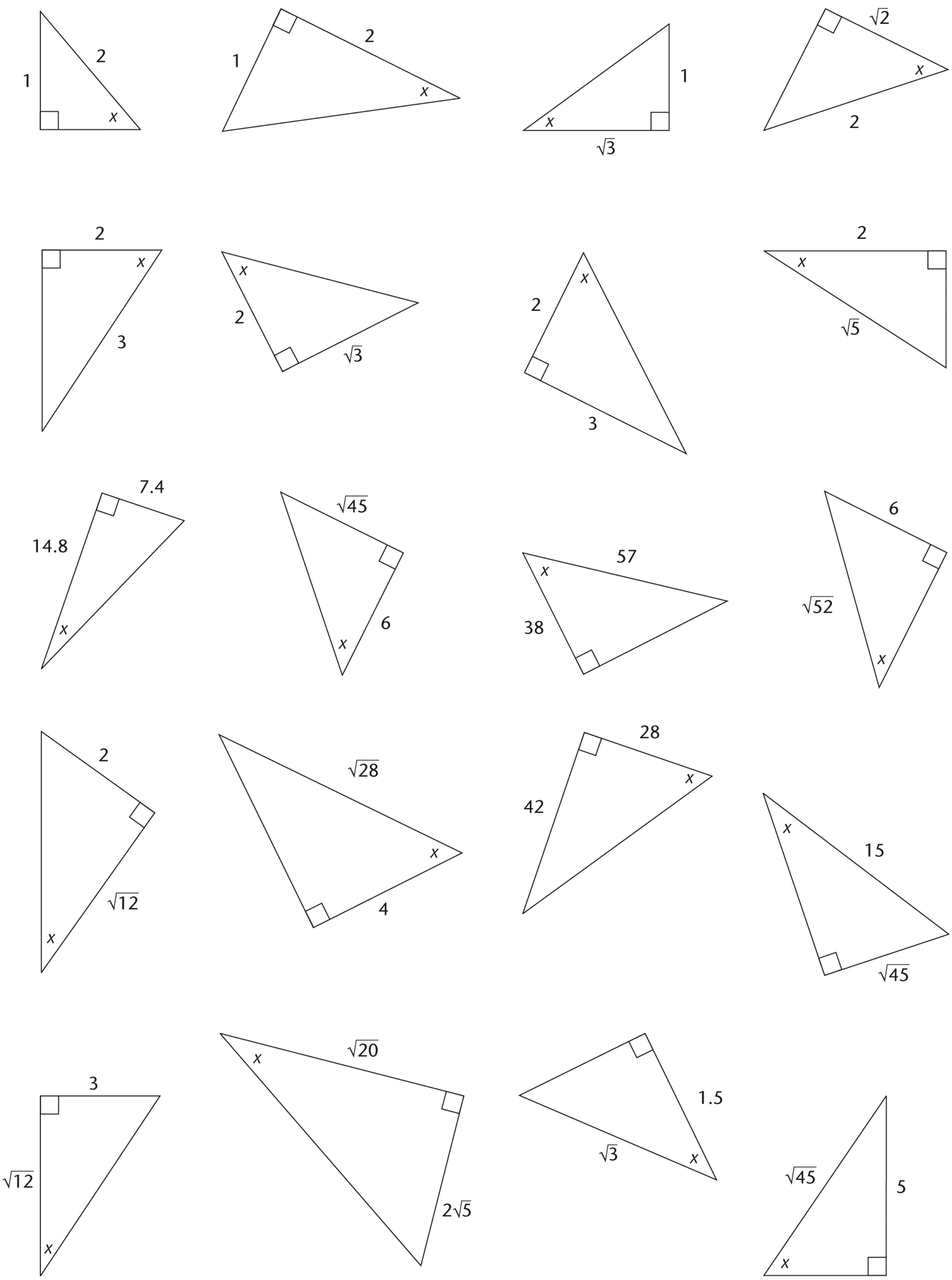
Resource 6: Enlargement grid



Resource 7: Photographic enlargement



Resource 8: Right-angled triangles for sorting



Resource 9: Trigonometric ratios for matching

$\sin x = \frac{1}{2}$	$\tan x = \frac{1}{2}$	$\cos x = \frac{2}{\sqrt{3}}$
$\cos x = \frac{3}{2}$	$\tan x = 1.5$	$\cos x = \frac{2}{\sqrt{5}}$
$\sin x = \frac{3}{\sqrt{13}}$	$\sin x = \frac{\sqrt{5}}{3}$	$\tan x = \frac{1}{\sqrt{3}}$
$\sin x = \frac{1}{\sqrt{5}}$	$\tan x = 1$	$\cos x = \frac{2}{\sqrt{7}}$
$\cos x = \frac{1}{\sqrt{2}}$	$\tan x = \frac{\sqrt{5}}{2}$	$\tan x = \frac{\sqrt{3}}{2}$

Resource 10: People geometry loci cards

Locus 1 The *mover* must walk, maintaining a position that is always a constant distance from the wall.
Locus 2 The *mover* must walk from a corner of the room, maintaining a position that is always always the same distance from both of the two walls that meet at the corner.
Locus 3 Position two people so that they are some distance apart. They must stand still. The *mover* must walk, maintaining a position that is always the same distance from both of the two stationary people.

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