Secondary mathematics guidance papers

Summer 2008

Department for Education

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department for
children, schools and families

Secondary mathematics guidance papers

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Introduction

This booklet comprises a set of guidance papers from the *Framework for secondary mathematics*. The guidance is intended to support mathematics departments in implementing the new programmes of study.

The papers address the following aspects:

1. Teaching mathematics so that pupils are able to see and make use of interconnections.
2. Ensuring that the range and content of mathematics is taught through the key processes.
3. Ensuring that pupils experience the full range of learning opportunities set out in the programmes of study, using teaching approaches that research has demonstrated are effective.

Making connections in mathematics

Although the strands of mathematics are described separately in the tables of objectives, there are, of course, many links between them. Mathematics is not a set of isolated topics or learning objectives but an interconnected web of ideas, and the connections need to be made explicit to pupils. Good planning ensures that mathematical ideas are presented in an interrelated way, not in isolation from each other. Awareness of the connections helps pupils to make sense of the subject, avoid misconceptions, and retain what they learn. So when you plan, aim to:

- present each topic as a whole, rather than as a fragmented progression of small steps, for example, show pupils that the place value system encompasses both whole and decimal numbers of any size, and that decimals and percentages are particular forms of fractions
- bring together related ideas across strands, for example, link work on metric measures to the decimal place value system, or link ratio and proportion in number to rates of change in algebra, to enlargement and similarity in geometry, and to proportional thinking in statistics and probability
- help pupils to appreciate that important mathematical ideas permeate different aspects of the subject, for example, the concepts of equivalence, inverse and order link the four number operations, are the key to manipulating algebraic expressions, and are central to the geometrical transformations of reflection, rotation, translation and enlargement
- use opportunities for generalisation, proof and problem-solving to help pupils to appreciate mathematics as a unified subject, for example, proof and mathematical argument involve chains of reasoning which link ideas together, so to prove that the sum of any three consecutive numbers will always be a multiple of 3 might involve forming algebraic expressions, using the distributive law to factorise an expression and appreciating the relationship between factors and multiples.

Working on the processes of mathematics will also be a unifying theme in your planning and teaching and will help pupils to see that developing these as transferable skills helps them to make progress. For example, you may plan specific points in the learning process at which you explicitly encourage pupils to consider alternative solutions as part of the process of communicating and reflecting. This process will be valid across all strands and will help pupils to see the unity of mathematics as a way of thinking and of solving problems.

Mathematical processes and applications

In the new programmes of study at Key Stages 3 and 4 there is a much greater emphasis on the key processes and attainment target 1 is different, now entitled 'Mathematical processes and applications'. This reflects the importance given to key processes in the 2008 curriculum. It also parallels the emphasis on key processes in other subjects and on aspects of process that reach across the curriculum, enshrined in the personal, learning and thinking skills (PLTS). A key aim of the curriculum is that pupils should see themselves more explicitly as learners and become aware of their developing skills, which are central to their work in school and to all aspects of their lives.

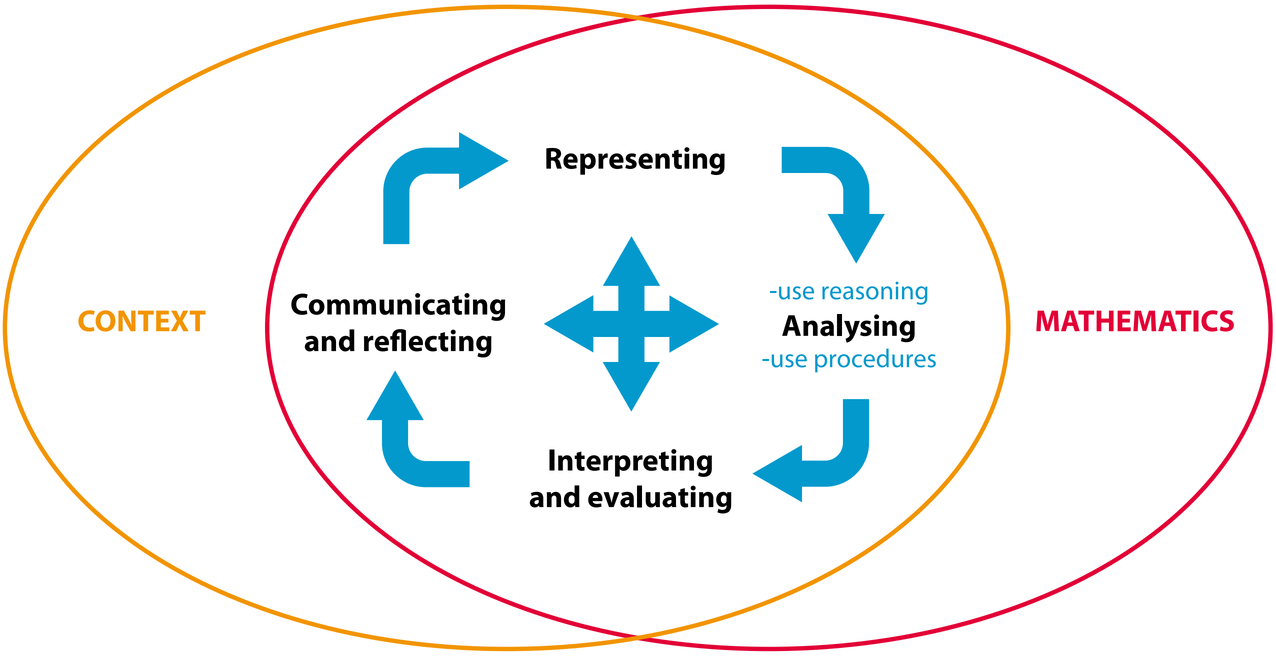
Using and applying mathematics was previously broadly described under the sub-headings of problem-solving, communicating and reasoning. Problem-solving lies at the heart of mathematics and involves a cycle of processes. These are elaborated in the key processes of the curriculum. By including mathematical procedures, well-defined routines and algorithms, a more complete description of process is achieved:

- Representing
- Analysing
  - use mathematical reasoning
  - use appropriate mathematical procedures
- Interpreting and evaluating
- Communicating and reflecting.

The process skills help pupils both to learn mathematics and to apply their mathematical subject knowledge to situations from life and the world of work. To ensure that they make progress in developing these skills and can function mathematically, pupils need to experience a rich 'diet' of applications that includes:

- increasingly **complex** applications, including non-routine or multi-step problems and extended enquiries, that require them to analyse a situation and sustain their thinking
- situations that are **unfamiliar** (in the sense that they are different from the context where the mathematics was developed), including applications to other subjects or aspects of their lives, and that require them to make connections and transfer their skills, sometimes in creative ways
- situations or problems that increase the **technical demand** of the mathematics required to solve them, including the application of more advanced concepts, more difficult procedures, or more rigorous argument and proof
- opportunities to develop greater **independence** and autonomy in problem-solving skills, so that they can select and apply a higher level of mathematics for themselves.

In summary, it is the context, and the mathematics to be applied to it, that determines the nature of the processing skills that pupils need and the level of challenge they face. It is helpful to think of a 'problem-solving cycle' but, as the diagram overleaf shows, many contexts require movement in and out of the cycle. For example, the 'representing' phase of a more complex problem may require some 'analysing', 'interpreting' or 'communicating' in order to set up the model.



Mathematical processes should be embedded within the everyday teaching of the strands of number, algebra, geometry and measures, and statistics, and in all cross-strand work. The related papers listed below give illustrative examples for each of the strands, including some observations of how ICT can be used to engage pupils with the key processes.

- Key processes in number
- Key processes in algebra
- Key processes in geometry and measures
- Key processes in statistics
- ICT and the key processes

Key processes in number

The foundations of the number system and processes of calculation are laid in Key Stages 1 and 2, and Key Stages 3 and 4 need to build on these foundations. Effective calculation is based on secure understanding of the number line, place value, relationships between operations and the laws of arithmetic (including the distributive law) and knowledge of number facts, together with strategies to quickly re-calculate any forgotten facts.

After calculation, the application of proportional reasoning is the most important aspect of elementary number. Proportionality underlies key aspects of number (ratios and rates, percentages and proportions), algebra (linearity, gradient), geometry and measures (enlargement, similarity, trigonometry) and statistics (proportions and probability). Being able to think multiplicatively as well as additively is a major shift that pupils need to make if they are not to remain dependent on informal approaches.

Representing

Representing a situation places it into a mathematical form that enables it to be worked on. This means more than just extracting the relevant numbers: it involves exploration of methods and tools and making choices between equivalent forms. Aspects of representing within number include:

- developing understanding of the number line and of tabular arrays, such as a place-value chart
- recognising the broader use of the equals sign to express the equivalence of two numbers or expressions
- choosing appropriate numerical representations, using knowledge of equivalent forms (e.g. of fractions, decimals and percentages)
- identifying the type of problem and the operations needed to reach a solution
- choosing between mental, written and calculator methods
- identifying the numerical data relevant to solving a problem and representing it to an appropriate degree of accuracy
- organising data in a suitable format, for example, a labelled 2×2 array representing a proportion
- breaking down more complex problems into a sequence of steps.

Simple problems are often solved informally, particularly where the numbers involved are easy to deal with mentally. Pupils also need to encounter more difficult, realistic and challenging problems, requiring methods that are more generally applicable, if they are to develop functional skills in number.

Analysing – use mathematical reasoning

Reasoning is involved when representing a numerical problem or situation, when calculating and when interpreting results. It includes:

- making and using connections between the number operations, for example, that indices represent repeated self-multiplication or that dividing by a number is the same as multiplying by its reciprocal
- making connections between fractions and division
- making connections between additive methods of proportional change and those using a single multiplier
- working logically to choose, use and review techniques to solve a multi-step problem
- using different techniques to analyse a situation, evaluate an approach or check a calculation, for example, looking at simpler cases or working backwards
- identifying, classifying and generalising numerical patterns.

Ultimately, reasoning in number will lead into algebra. As the two strands develop in tandem, pupils are in a position to realise that explaining and convincing within number becomes a constraint and that proving and generalising requires a symbolic form and the discipline and structure of algebra.

Analysing – use appropriate mathematical procedures

Using appropriate procedures involves manipulating numbers, using and applying algorithms and techniques, and monitoring the accuracy of methods and solutions.
Appropriate procedures in number include:

- building on the approach to calculation developed in Key Stages 1 and 2, which emphasises mental methods and written methods that are gradually refined and made efficient
- visualising images, such as a number line or other representation, to support mental methods
- extending procedures to calculations with fractions, decimals and percentages
- developing efficient use of calculators and a spreadsheet (e.g. for optimisation problems)
- developing multiplicative methods for solving problems involving direct proportion, extending to inverse proportion
- developing routines for estimating, approximating and checking calculations.

When entering Key Stage 3, pupils need opportunities to discuss their mental methods of calculation, to explore and evaluate different approaches to particular calculations, and to practise and extend their mental fluency. They also need opportunities to discuss their written methods of calculation, including expanded methods. When refining methods (e.g. developing column procedures), they should do so with consideration for understanding and accuracy as well as efficiency. They should learn when it is appropriate to use a calculator and how to use a basic calculator, and later a scientific calculator, accurately and efficiently. Mental methods of approximating, estimating and checking are an important adjunct to all calculation.

Interpreting and evaluating

Aspects of interpreting and evaluating in number include:

- considering the appropriateness and accuracy of numerical results
- relating numerical results to the problem or context under consideration and drawing conclusions
- considering the effectiveness of different strategies, for example, when completing a mental calculation

- following the logic of someone else's calculation, particularly the effects of rounding
- appreciating when to move to algebraic form rather than elaborating numerical examples.

Communicating and reflecting

Aspects of communicating and reflecting of particular relevance to number include:

- using precise language and symbolism to set out solutions in an organised and understandable way
- discussing and reflecting on different approaches to solving a numerical problem
- linking mental methods with jottings and representations, for example, on a number line
- comparing the efficiency of calculation procedures
- considering the effects of rounding.

Resources for number

A range of resources to support the development of key processes within number are included in the 'Ideas for rich tasks' folder within the *Secondary mathematics planning toolkit*:

- *Interacting with mathematics in Key Stage 3 – Proportional reasoning:*
  - *Year 7 fractions and ratio*
  - *Year 8 multiplicative relationships*
  - *Year 9 proportional reasoning*
  - *Enhancing PR in Y8 and Y9*
  - *PR transition lessons*
- *Teaching mental mathematics from level 5:*
  - *Number*
  - *Measures and mensuration in number*
- Standards Unit *Improving learning in mathematics:*
  - *Mostly number (Sessions N1–N13).*

Key processes in algebra

Algebra in Key Stages 3 and 4 is based on the generalisations of relationships familiar from basic number. It is developed to include the use of equations, formulae and identities, and sequences, functions and graphs. Algebra is purposeful when pupils encounter sufficiently complex situations where objects or relationships require representation in symbolic or graphical form. These occur frequently when describing generalisations underlying particular relationships. To use and make sense of algebra, pupils need opportunities to relate it to their knowledge of the arithmetical operations. Suitable contexts for algebraic representation may come from within mathematics (for example, exploring number patterns and puzzles, or finding areas of shapes), by linking with other subjects or from real-life applications. It should include use of ICT, such as graph-plotting and spreadsheet software to explore functions.

Representing

Representing a situation places it into a mathematical form that enables it to be worked on. In algebra this might mean trying out and choosing between different diagrammatic, graphical and symbolic forms arising from looking at the problem or situation from different points of view. Aspects of representing within algebra include:

- identifying assumptions, variables and relationships in order to create a mathematical model
- developing understanding of algebraic conventions; for example, conventions of writing terms and expressions, coordinate points and equations of lines, vectors and magnitude of vectors
- constructing algebraic expressions, equations, formulae and identities; for example, understanding and using signs such as $=$, $\neq$, $<$ and $>$ to represent relationships between variables
- choosing appropriate algebraic representation of such relationships, using knowledge of equivalence forms (e.g. tables, functions and graphs), so that the context can be analysed and the solution communicated
- choosing the tools most appropriate to represent the mathematics drawn from the situation; for example, a graphical calculator or a spreadsheet.

As well as giving point to the subject, experience of algebraic representation is crucial if pupils are to understand and use precise algebraic language. Giving explicit attention to this helps them to understand the conventions for using letter symbols and constructing algebraic expressions. It can also give pupils insights into algebraic structure and order of operations, needed when transforming or interpreting symbolic and graphical representations.

Analysing – use mathematical reasoning

Algebra as a tool lies at the heart of much mathematical reasoning. Pupils need opportunities to experience the power of algebra in expressing generality. This includes:

- identifying and describing numerical patterns and relationships, both symbolically and graphically
- making connections with arithmetical operations and with equivalent algebraic forms when transforming expressions and equations
- making connections between sequences, functions and graphs, and exploring the effects of varying values
- making generalisations, explaining and proving, and relating results to the context of the problem.

Analysing – use appropriate mathematical procedures

Using appropriate procedures involves manipulating expressions, equations and graphs, using and applying techniques and accurate notation, and monitoring the accuracy of methods and solutions. Appropriate procedures in algebra include:

- generating equivalent expressions and equations, including a simplified form
- factorising and expanding expressions and equations
- solving equations exactly and approximately
- manipulating formulae, including changing the subject of the formula
- substituting values into equations and formulae; for example, evaluating a formula to convert temperature in degrees C to degrees F.

Algebra in Key Stages 3 and 4 is generalised arithmetic. It requires understanding of the commutative, associative and distributive laws as they apply to the number operations, and of relationships between operations, including inverses. Pupils can be supported to generalise the rules with letters in place of numbers; for example, $ab = c$ implies:

$$ba = c \qquad b = \frac{c}{a} \qquad a = \frac{c}{b} \qquad 2ab = 2c \qquad 2ab + 1 = 2c + 1\ldots$$

Taking an exploratory approach to transforming algebraic expressions and equations, where pupils are regularly asked to write expressions in different ways ('find as many ways as you can'), builds their algebraic skills. Pupils:

- gain confidence in manipulating expressions into different equivalent forms
- gain insights into which of a range of possible transformations will be both valid and efficient as a next step, for example in solving an equation or rearranging a formula
- develop increasing fluency with algebraic manipulation without being rule-bound and, when the steps in a procedure are not obvious, are able to resolve difficulties for themselves.

Interpreting and evaluating

Aspects of interpreting and evaluating in algebra include:

- relating numerical results, such as the solution of an equation, to the context under consideration
- interpreting general statements or conclusions expressed in algebraic form (e.g. an expression or formula) and considering their significance
- recognising the difference between numerical evidence and algebraic proof
- interpreting graphs and graphical features such as points of intersection, gradients and the general shape of a graph
- evaluating different approaches; for example, where another pupil has represented the problem or approached its solution in a different way.

Communicating and reflecting

Aspects of communicating and reflecting in algebra include:

- recognising and using the fact that algebraic language (symbolic and graphical) is a powerful form of communication for expressing the steps in an argument or the conclusions of an enquiry
- considering alternative approaches; for example, comparing algebraic, graphical and numerical approaches to tackling a problem
- making links to related problems or to different problems with a similar structure.

Resources for algebra

A range of resources to support the development of key processes within algebra are included in the 'Ideas for rich tasks' folder within the *Secondary mathematics planning toolkit*:

- *Interacting with mathematics in Key Stage 3 – Algebra:*
  - *Constructing and solving linear equations* (Y7, Y8 and Y9 booklets)
- *Teaching mental mathematics from level 5:*
  - *Algebra*
  - *Measures and mensuration in algebra*
- Standards Unit *Improving learning in mathematics:*
  - *Mostly algebra* (sessions A1–A14)

Key processes in geometry and measures

Geometry in Key Stages 3 and 4 is the study of points, lines and planes and the shapes that they can make, together with the study of plane transformations. A key aspect is the use and development of deductive reasoning in geometric contexts. Geometrical activities can be linked to drawing, construction and loci, and work on measures and mensuration. By ensuring that pupils have a range of suitable experiences you can develop their knowledge and understanding of shape and space and their appreciation of the ways that properties of shapes enrich our culture and environment.

Representing

Developing a mental facility to recognise geometric features is an important skill which underpins all geometric representation. In order to represent a situation geometrically, pupils have to make assumptions about the major features which are important to the context and then decide how to present these features in their 'mind's eye', on paper, on a screen or through a physical model. Pupils need to be given the opportunity to explore problems, trying out different representations and choosing between them. This will involve:

- visualising and estimating
- sketching
- constructing accurate diagrams
- building models, both physical and virtual
- representing in algebraic or numerical form, including appropriate use of units.

Contexts need to include those in which pupils work with both static and dynamic images, including 2-D images of 3-D objects, and should make use of appropriate ICT packages.

Analysing – use mathematical reasoning

Geometrical reasoning makes a distinctive contribution to mathematics and its applications. Pupils need opportunities to develop language with which to describe what they see and to explain their thinking. They engage with geometrical reasoning when they:

- visualise and work with images, explaining relationships within an image; for example, transforming an image and identifying what changes and what stays the same
- identify and classify geometrical patterns, including symmetrical designs in nature or from human artefacts
- form simple assumptions (e.g. about points and lines), developing short chains of reasoning to deduce properties of shapes, such as the angle sum of a triangle.

Analysing – use appropriate mathematical procedures

Procedures that pupils will use when working on geometrical situations involve manipulating geometric images, using and applying techniques and accurate notation, and measurements. This will include:

- visualising points, lines, shapes and solids, including changes of perspective and dynamic images
- sketching points, lines, shapes and solids, including nets of solids and the use of geometric conventions

- drawing accurately annotated diagrams (on paper and on screen), including elevations, cross-sections and scale drawings
- building models of 3-D objects, including those interpreted from 2-D drawings
- calculating and estimating lengths, areas, volumes and angles using appropriate formulae, when solving problems involving measures.

Interpreting and evaluating

Aspects of interpreting and evaluating in geometry and measures include:

- interpreting features of a diagram or other representation and relating those features to the context or situation represented
- considering the appropriateness and accuracy of measurements and consistency of units
- evaluating results or conclusions to a geometrical problem
- appreciating the difference between evidence from particular cases and the generality of geometrical proof.

Communicating and reflecting

Aspects of communicating and reflecting in geometry and measures include:

- using diagrams to communicate findings effectively
- setting out the steps of a deductive argument in a logical order, using precise language and symbolism to give clear reasons for each step
- making connections between deduced properties and definitions of shapes
- considering alternative solutions and identifying commonalities such as equivalent stages of a solution
- choosing from a range of forms through which to communicate solutions; for example, between diagrams and symbols, physical models, static and dynamic images, scale drawing and sketching.

Resources for geometry and measures

A range of resources to support the development of key processes within geometry and measures are included in the 'Ideas for rich tasks' folder within the *Secondary mathematics planning toolkit*:

- *Interacting with mathematics in Key Stage 3 – Geometrical reasoning:*
  - *Y9 geometrical reasoning* (mini-pack, resource sheets and problem bank)
- *Teaching mental mathematics from level 5:*
  - *Shape and space*
  - *Measures and mensuration in shape and space*
- Standards Unit *Improving learning in mathematics:*
  - *Mostly shape and space* (Sessions SS1–SS8)
- *ICT in mathematics:*
  - *Mathematics with ICT in Key Stage 3: Geometry lessons*

Key processes in statistics

Statistics, the mathematics of making inferences from data, is best taught in a coherent way in the context of real statistical enquiries so that learning arises naturally from the whole handling data cycle (see below). As an enquiry develops you will need to reinforce and develop certain skills and concepts by direct teaching of particular objectives. It is easier to make sure that problems are relevant if at least some of the enquiries are linked to other subjects. ICT is important for representation and analysis of data. It also enables simulation of events and the processing of large data sets, which helps pupils to understand the relationship between experimental and theoretical probability.

The handling data cycle remains valid and is closely linked to the key processes:

- specifying the problem and planning (representing)
- collecting data (representing and analysing)
- processing and representing the data (representing and analysing)
- interpreting and discussing the results (interpreting and communicating).

Representing

In a statistical enquiry, representing is part of almost all elements of the handling data cycle. It involves:

- suggesting a problem to consider using statistical methods, framing questions and raising conjectures
- deciding what data are relevant and identifying primary or secondary sources
- designing ways of capturing the required data, including minimising sources of bias
- creating representations of the data, including the use of ICT, for example, tabulation, grouping, arrays, diagrams and graphs.

If pupils can represent data as part of a statistical enquiry then they are better positioned to become responsible citizens who can select and sift information thoughtfully and use mathematics with confidence to inform decision-making. Help pupils to see that representation is a major focus of statistics and that it is important to tie together the decisions they make at the different stages of the handling data cycle.

Analysing – use mathematical reasoning

Mathematical reasoning is required at all stages of the handling data cycle:

- when specifying and planning by working logically, identifying constraints and considering available techniques; also by exploring conjectures and using knowledge of related problems
- when collecting data by working systematically, exploring the effects of varying values in situations where there is random or systematic variation
- when processing and representing data, making connections within mathematics and identifying patterns and relationships, and making use of feedback from different audiences
- when interpreting and discussing results, explaining and justifying inferences drawn from the data, recognising the limitations of any constraints or assumptions made; using feedback to re-assess initial conjectures and adjust aspects of the handling data cycle.

Analysing – use appropriate mathematical procedures

Using appropriate procedures involves manipulating data into suitable forms for accurate representation, calculation and communication. This will involve monitoring the accuracy of methods and solutions. Appropriate procedures in a statistical enquiry are:

- using systematic methods for collecting data from primary and secondary sources
- constructing tables, graphs and diagrams to present data in an organised form
- calculating summary statistics, for example, measures of average and spread
- calculating experimental and theoretical probabilities.

Interpreting and evaluating

Interpreting and evaluating results is fundamental to any statistical enquiry. It includes:

- interpreting tables, graphs and diagrams, and drawing inferences to support or cast doubt on initial conjectures
- interpreting probabilities when assessing the likelihood of a particular outcome
- comparing distributions and making inferences
- looking at data to find patterns and exceptions
- considering the effects of changes to the data (e.g. removing outliers, adding items, making proportional changes)
- appreciating why the interpretations placed on data have a degree of uncertainty and can be misleading
- appreciating convincing arguments, but knowing that these do not constitute proof.

Communicating and reflecting

Effective communication and reflection is of particular relevance in statistics. It includes:

- preparing a brief report of a statistical enquiry, using tables, graphs and diagrams to summarise data and support interpretations and inferences drawn from the data
- using precise language to summarise key features pertinent to the conjectures raised
- presenting support for conclusions in a range of convincing forms
- presenting a balanced conclusion where results are not convincing
- considering alternative approaches if results do not provide sufficient evidence.

Resources for statistics

A range of resources to support the development of key processes within statistics are included in the 'Ideas for rich tasks' folder within the Secondary mathematics planning toolkit:

- *Interacting with mathematics in Key Stage 3 – Handling data:*
  - *Y8 handling data mini-pack*
  - *'Wise words and other tasks'* (Securing progression in handling data)
- *Teaching mental mathematics from level 5:*
  - Published 2009
- Standards Unit *Improving learning in mathematics:*
  - *Mostly statistics* (Sessions S1–S8)

ICT and the key processes

The term 'ICT' as applied to the teaching of mathematics includes a range of hardware (including calculators), and general and subject-specific software. Many mathematics classrooms now have interactive whiteboards which allow daily opportunities for ICT to be used in the classroom, both as a demonstration tool for teachers and as a way of allowing pupils to interact with software. The opportunities that ICT provides mean that teachers can reconsider the most appropriate ways of teaching mathematics. Concepts, structures and processes can be represented in new and revealing ways, often using dynamic images, which permit insights and understandings that were difficult to convey previously.

ICT should be considered an essential tool for both teachers and pupils. It can increase pace in lessons, improve pupil engagement and encourage discussion. Many teachers report that using ICT has led to pupils showing greater confidence in their mathematics, often shown by pupils 'having a go' instead of giving up when they meet a challenge.

The following illustrates just some of the ways that ICT can be used to support learning through each of the key processes.

Representing

ICT offers powerful forms of representation, including on-screen images and computer models in various forms. Examples of using ICT to represent include:

- choosing an appropriate computer representation of a problem, trying out ideas and experimenting
- identifying variables to be considered in the computer model, for example, the parameters in an equation or probabilities in a statistical simulation
- selecting the mathematical data that needs to be entered into the computer.

Analysing – use mathematical reasoning

Computers have a number of features that can be exploited to foster pupils' ability to reason. These include:

- making connections, for example, between an equation, a table and a graph
- visualising and working with dynamic images, for example, using dynamic geometry software to explore the properties of lines, angles and shapes
- making conjectures and generalisations fostered, for example, by use of the 'hide and reveal' facility of a computer
- exploring the effects of varying values, for example, in a spreadsheet or statistical model, posing the question 'what happens if…?'
- taking account of feedback from a computer, particularly exploiting its rapid processing capability.

Analysing – use appropriate mathematical procedures

Calculators and computers are themselves procedural devices, but the user also has to learn appropriate procedures in order to make effective use of them.

Examples include:

- procedures for constructing and manipulating graphs, charts, diagrams and other screen images
- routines for using a basic, scientific or graphical calculator accurately and efficiently, including, where appropriate, trial and improvement methods
- use of correct notation and computer syntax.

Interpreting and evaluating

Setting up a model and using the processing power of the computer necessitates particular skills in interpreting and evaluating the output. Some of the skills required include:

- determining whether the output is in an appropriate and useful form
- interpreting the meaning of the output in terms of the situation being modelled
- looking for patterns and correlations (e.g. in a spreadsheet) and evaluating their significance
- drawing conclusions and constructing convincing general statements
- being aware of the strength or weakness of empirical evidence gathered when using ICT.

Communicating and reflecting

ICT offers ways of communicating in mathematics. For example:

- organising findings and using graphic and dynamic forms of presentation
- discussing computer outputs, for example, the effect of changing an input or the value of a variable
- comparing alternative approaches, for example, an iterative method with an analytical approach.

A range of resources to support ICT in mathematics are included in the 'Ideas for rich tasks' folder within the *Secondary mathematics planning toolkit*.

Teaching and learning approaches

Guidance on teaching and learning approaches is presented in three sections:

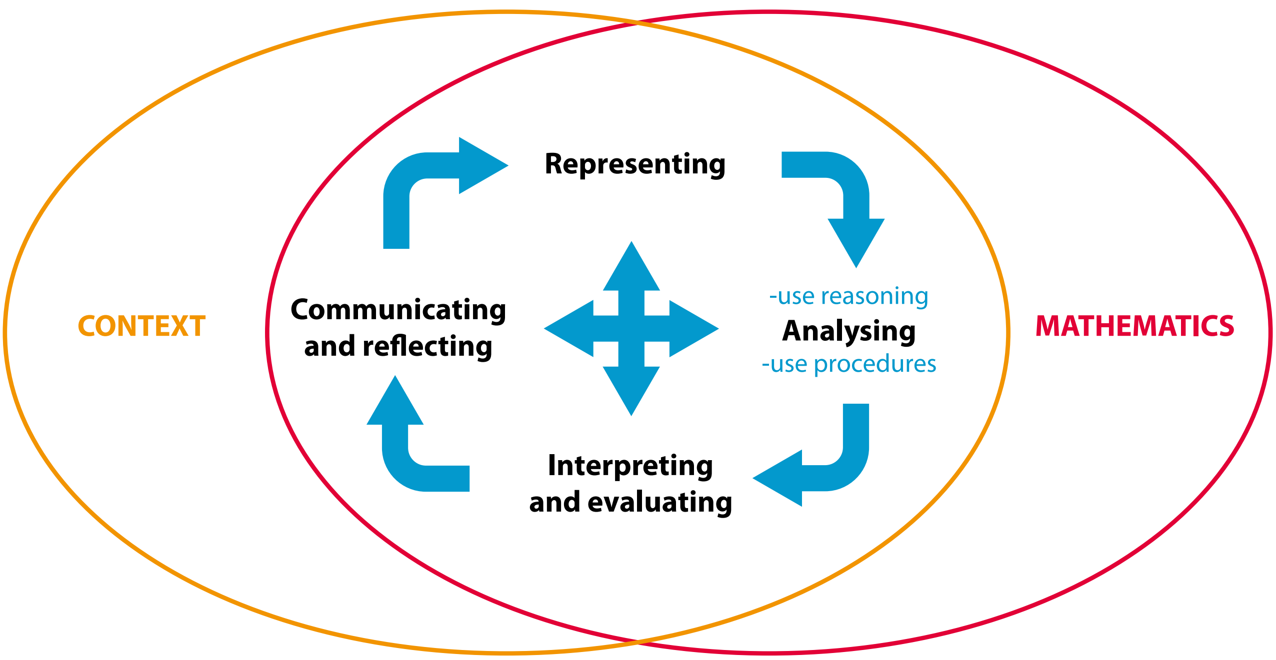
- 'Some principles for effective learning', based on the 2008 programme of study in mathematics
- 'Some principles for effective teaching', based on research over many years into the teaching of mathematics
- 'Further support to develop pedagogy and practice', which references existing Strategy guidance on lesson design, teaching repertoire, etc.

Some principles for effective learning

This section is informed by the curriculum aims of the 2008 programme of study. By synthesising and interpreting the aims, key processes, key concepts and curriculum opportunities, the intention is to provide a supportive reference paper which the whole department can use to reflect on priorities for development in teaching and learning, and so phase the implementation of the new programme of study.

Pupils learn about and learn through the key mathematical processes

Key processes need to be experienced as components of a whole cycle. This can be reflected within a single lesson as well as through a unit of work. Investigative and problem-solving opportunities should be designed so that pupils cycle around the processes several times and also move backwards and forwards between the stages as ideas mature, modify and change. In this way the notion of a cycle provides a helpful structure but does not become restrictive.



The diagram represents the dual nature of mathematics: it is both a tool for solving problems in a wide range of contexts and a discipline with a distinctive and rigorous structure.

Pupils become successful learners by developing competence in applying mathematics effectively in a range of contexts, including those from within mathematics itself. There are two ways of thinking about pupils' experience of the key mathematical processes which lie at the heart of the revised programme of study.

• They need opportunities to **learn about** the mathematical processes and to reflect on how they are improving in these skills. This could include designing lessons or units where there is no new content and the focus is on improving the process skills.	• They need opportunities to **learn through** using the mathematical processes. As pupils gain confidence in the skills of applying these processes, they can use them to develop their understanding of topics within the range and content of the curriculum.

Pupils work collaboratively and engage in mathematical talk

It is through paired and group work that pupils gain confidence in their ability to communicate mathematics effectively. Choosing a rich task will usually provide pupils with the chance to explain and justify, question and disagree. Over time, the level of dialogue in the classroom becomes more mathematically rich as pupils pose questions to each other and develop more convincing arguments orally. As this kind of dialogue becomes a regular part of their work on mathematics pupils are forced to think in this way, preparing questions for one another and rehearsing arguments. We could describe this as developing a habit of 'self talk'; that is, they are naturally developing the thinking that will support more independent work.

Pupils work on sequences of tasks

When planning and teaching of units of work, there need to be sequences of lessons which do not move too quickly from one topic to another or from one task to another. Instead, pupils need to be provided with a sequence of learning which is planned to become more challenging within a phase of a unit. One way of doing this is to select a task, or sequence of related tasks, which develop over a number of lessons. This has the advantage of reducing the burden of producing and introducing different tasks in each lesson. It means that more of a lesson is dedicated to pupils actively doing mathematics rather than listening to instructions for new and different topics and tasks. Most importantly, a sequence of tasks, involving the same mathematics in increasingly difficult or unfamiliar contexts, or increasingly demanding mathematics in similar contexts, makes mathematical progression more explicit to the pupils. In this way pupils develop the competence to apply suitable mathematics accurately within the classroom and beyond.

Pupils select the mathematics to use

Pupils can begin to see the power and purpose of their mathematical learning when they are given the opportunity to make decisions about the mathematical tools (including ICT) available to help them solve a problem or investigate from a given stimulus. As pupils use existing mathematical knowledge to investigate or create solutions to unfamiliar problems, their confidence increases and they come to see that doing mathematics is an interesting and enjoyable activity. They are then more likely to effectively apply mathematical skills in life, in their wider studies and ultimately in employment. Unit plans adapted to meet the new curriculum should build in a variety of open and closed tasks, ensuring that the contexts for some tasks are real and those for others are abstract. In this way pupils come to appreciate mathematics for itself, as well as understanding that it is used as a tool in a wide range of contexts.

Pupils tackle relevant contexts beyond the mathematics classroom

In order for pupils to be functional in mathematics and motivated to take their learning further, they need to hone their knowledge, skills and understanding by applying suitable mathematics accurately within the classroom and beyond. This means planning units where pupils are not learning new content but are working on problems that arise in other subjects and in contexts beyond the school, such as architecture or engineering. In many cases a solution will involve using a mathematical model to interpret or represent situations. Applications involving modelling changes in society and the environment or managing risk (e.g. insurance and investments) could be used to stimulate discussion about important issues such as financial capability or environmental dilemmas. The assumptions and simplifications involved in the process of modelling a real-life context should be made explicit so that pupils come to realise that mathematics itself is essentially abstract and that a model or representation has limitations to its scope.

Pupils are exposed to the historical and cultural roots of mathematics

If pupils are given the chance to learn about problems from the past that led to the development of particular areas of mathematics, they can begin to appreciate that people of all cultures use mathematics to make sense of the world around them. They may be fascinated to find out that pure mathematical findings sometimes precede practical applications, and their curiosity may be aroused when they learn that mathematics continues to develop and evolve. This will engage and motivate pupils to become more aware of the nature of mathematics and of the mathematics around them.

Some principles for effective teaching

Research shows that the following principles underlie effective teaching. They are based on those included in *Improving learning in mathematics: challenges and strategies*, by Malcolm Swan of the University of Nottingham, in the 'Standards Unit box' (Swan, M. (2005) *Improving learning in mathematics,* DfES Standards Units, Ref: 1599-2005DOC-EN). The list is provided to support evaluation of current teaching approaches and to stimulate departmental discussions around improving the effectiveness of current teaching.

Build on the knowledge pupils bring to a sequence of lessons

Design activities which uncover prior learning and offer pupils opportunities to express their understanding. For example:

- pose a problem to the whole class to stimulate paired discussion and to set the agenda for the next few lessons
- set up pairs or groups to draw and share a concept map or equivalent diagram showing the interconnections of existing understanding. Revisit the 'maps' and add to them as the learning emerges throughout the unit.

The following Strategy resources in the 'Rich tasks' folder of the *Secondary mathematics planning toolkit* may help with this approach:

- *Leading in learning* (KS3 and KS4 training manuals and exemplification in mathematics)
- *Bridging plans from KS3 to KS4: Mathematics*
- *Interacting with mathematics in Key Stage 3 – proportional reasoning:*
  - *Year 7 fractions and ratio* (Mini-pack and resources, especially the key lesson).

Expose and discuss common misconceptions

Pupils make mistakes for a variety of reasons. Errors are often caused by lapses in concentration, hasty reasoning, memory overload or a failure to notice important features of a problem. Others, however, are symptoms of more profound mathematical difficulties. Where mistakes are the result of consistent, alternative interpretations of mathematical ideas, we refer to them as misconceptions. These should not be dismissed as 'wrong thinking' as they may be necessary stages of conceptual development. Design activities so that misconceptions are systematically exposed by allowing time in the lessons for pupils to reflect and discuss these conflicts. For example:

- ask pupils to complete a task using more than one method and then to resolve conflicting answers
- present statements to be classified and justified as always true, sometimes true or never true.

The following Strategy resources in the *Secondary mathematics planning toolkit* may help with this approach:

- *Misconceptions in mathematics* (Pedagogy folder, Improving subject knowledge sub-folder)
- *Teaching mental mathematics from level 5:*
  - *Shape and space* (Rich tasks folder)

You might also find useful, as a separately available CD-ROM including video:

- *Mathematics: developing dialogue and reasoning* (Ref: 00243-2006CDO-EN)

Develop effective questioning

Aim to invite a range of responses to your questions by asking more open and/or probing questions which promote higher-level, reflective thinking. Allow time for pupils to think before offering help or moving on to ask someone else, and allow time for yourself to think about your response. For example:

- establish a routine through which pupils share their answers in pairs before you take whole-class feedback. This 'pair/share' not only builds confidence but also increases the number of pupils who feel that their response has been heard.
- be explicit about the types of questions you use and encourage the pupils to use the same types of question. A display of question stems can be helpful for you and the pupils:
  - *What if…?*
  - *Why do you think…?*
  - *When would…not work?*
  - *How do you know…?*

The following Strategy resources in the *Secondary mathematics planning toolkit* may help with this approach:

- Standards Unit *Improving learning in mathematics* (Pedagogy folder)
- *Pedagogy and practice: Teaching and learning in secondary schools*
  - *Unit 7: Questioning* (Pedagogy folder)

Also, available as separate CD-ROMs:

- *Assessing pupils' progress in mathematics at Key Stage 3* (Probing questions)
- *Mathematics: developing dialogue and reasoning* (Ref: 00243-2006CDO-EN)

Use cooperative small-group work

Ensure that everyone is confident and benefits from participating in discussions by designing tasks which require collaboration in pairs or small groups, and establish this as a regular feature of mathematics lessons. For example:

- use classification activities with only one set of objects per pair or group so that joint decisions have to be made
- ask pupils to create a spider diagram of connections with one large sheet of paper for three pupils

The following Strategy resources in the *Secondary mathematics planning toolkit* may help with this approach:

- *Pedagogy and practice: Teaching and learning in secondary schools:*
  - *Unit 10: Group work* (Pedagogy folder)
- *Interacting with mathematics in Key Stage 3 – Handling data*
  - *Y8 handling data minipack* (Rich tasks folder)

Emphasise methods rather than answers

Focus on pupils developing their repertoire of appropriate methods, rather than on obtaining correct answers to a long list of similar problems. This is likely to involve aiming to work on fewer problems than would appear in a typical textbook exercise. For example:

- direct pupils to tackle the same problem using more than one method and work in pairs to compare solutions and evaluate their efficiency
- ask pupils redesign a problem so that it is more challenging or simpler and give it to their group to solve.

The following Strategy resources in the *Secondary mathematics planning toolkit* may help with this approach:

- *Interacting with mathematics in Key Stage 3* (Rich tasks folder):
  - *Handling data ('Wise words and other tasks')*
  - *Proportional reasoning (Y8 multiplicative relationships, Y9 proportional reasoning, Enhancing PR in Y8 and Y9)*

Use rich collaborative tasks

Think about how to design tasks that motivate a need to learn and encourage pupils to be creative, curious and reflective. Pupils' mathematical thinking will be improved if they have to make decisions and ask questions. Learning is made memorable when pupils enjoy the tasks and are surprised by outcomes. Richer tasks allow all learners to find something challenging and at an appropriate level to work on. Examples of accessible and extendable tasks can be developed from 'routine' tasks by changing the initial stimulus and the questions asked. For example:

- help pupils to consolidate their understanding of algebraic factorisation, expansion and simplification by working with jigsaw/domino cards showing matching expressions; extend this to include their own design of such cards
- ask two groups of pupils to debate opposing arguments which support or refute a hypothesis where data is supplied in a spreadsheet.

The following resources in the *Secondary mathematics planning toolkit* may help with this approach:

- *Teaching mental mathematics from level 5* (Rich tasks folder)
- Standards Unit *Improving learning in mathematics* (Rich tasks folder)

Create connections between mathematical topics

Design activities for existing units that make explicit connections within and across mathematical topics. For example:

- matching tasks which require pupils to recognise different representations of the same mathematical idea.

You may also plan to include more cross-strand units to develop stronger connections. For example:

- a functional mathematics unit presenting a real-life context requiring exploration or investigation; in such units pupils could work on a range of mathematical connections needed to reach a resolution.

The following resources in the 'Rich tasks' folder of the *Secondary mathematics planning toolkit* may help with this approach:

- Standards Unit *Improving learning in mathematics*
- *Teaching mental mathematics from level 5:*
  - *Measures and mensuration* (Set of booklets)
- *Interacting with mathematics in Key Stage 3 – Proportional reasoning:*
  - *Y9 proportional reasoning*

Use technology in appropriate ways

Present mathematical concepts in dynamic, visually exciting ways that engage and motivate learners. Introduce, explore and represent concepts, structures and processes in new and revealing ways. Often, dynamic images will permit insights and understandings that are difficult to convey in other ways. For example:

- display an equation of the form $y = mx + c$ on the same screen as the associated table and graph in order to explore the relationship between them
- explore a dynamic diagram showing how the angle formed between two straight lines changes as the lines move; extend to parallel lines and an intercept.

The following Strategy resources in the 'Rich tasks' folder of the *Secondary mathematics planning toolkit* may help with this approach:

- *Interacting with mathematics in Key Stage 3 – Proportional reasoning:*
  - *Y7 fractions and ratio* (Interactive teaching programmes)
- *ICT in mathematics* (ICT lesson plans)

Further support to develop pedagogy and practice

The 2001 *Framework for teaching mathematics: Years 7, 8 and 9* established several principles which teachers found useful in guiding their planning. They included dimensions of a teaching repertoire such as modelling, questioning and explaining, and aspects of lesson design such as structuring learning and starters and plenaries.

To further support these developments the *Pedagogy and Practice* materials were published a few years later.

Pedagogy and Practice: Teaching and learning in secondary schools is often referred to as 'the ped pack'. It is a suite of study guides created to support the professional development of teachers across all subjects in secondary schools. They provide guidance on the relationship between pedagogic approaches (teaching models), teaching strategies, techniques and methods of creating the conditions for learning in order to inform lesson design. The techniques suggested in each study guide are tried and tested, and draw on both academic research and the experience of practising teachers.

Many teachers, who began to work on new teaching strategies from the initial, brief guidance in the Framework, moved on to more detailed developments through the 'ped pack' guidance. For example, the booklet on structuring learning elaborates the original Framework guidance on structured lessons. It describes dividing lessons into a series of episodes, choosing from a range of strategies and techniques to motivate pupils, and examines three pedagogic approaches (direct interactive, inductive and exploratory) to show how they can help pupils develop tools for learning, such as inductive thinking or enquiry skills.

The full list of booklets is given below, but you are unlikely to require the entire set at any one time. Instead, think about what support you need and consider downloading one or two booklets; most are only 24 pages long and they can all be accessed at (www.standards.dfes.gov.uk/secondary/keystage3/all/respub/sec_pptl0).

Designing lessons

Unit 1: Structuring learning

Unit 2: Teaching models

Unit 3: Lesson design for lower attainers

Unit 4: Lesson design for inclusion

Unit 5: Starters and plenaries

Teaching repertoire

Unit 6: Modelling

Unit 7: Questioning

Unit 8: Explaining

Unit 9: Guided learning

Unit 10: Group work

Unit 11: Active engagement techniques

Creating effective learners

Unit 12: Assessment for learning

Unit 13: Developing reading

Unit 14: Developing writing

Unit 15: Using ICT to enhance learning

Unit 16: Leading in learning

Unit 17: Developing effective learners

Creating conditions for learning

Unit 18: Improving the climate for learning

Unit 19: Learning styles

Unit 20: Classroom management

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