Department for Education

Further mathematics

AS and A level content

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Content for further mathematics AS and A level for teaching from 2017

Introduction

1. AS and A level subject content sets out the knowledge, understanding and skills common to all specifications in further mathematics.

Purpose

2. Further mathematics is designed for students with an enthusiasm for mathematics, many of whom will go on to degrees in mathematics, engineering, the sciences and economics.

3. The qualification is both deeper and broader than A level mathematics. AS and A level further mathematics build from GCSE level and AS and A level mathematics. As well as building on algebra and calculus introduced in A level mathematics, the A level further mathematics core content introduces complex numbers and matrices, fundamental mathematical ideas with wide applications in mathematics, engineering, physical sciences and computing. The non-core content includes different options that can enable students to specialise in areas of mathematics that are particularly relevant to their interests and future aspirations. A level further mathematics prepares students for further study and employment in highly mathematical disciplines that require knowledge and understanding of sophisticated mathematical ideas and techniques.

4. AS further mathematics, which can be co-taught with A level further mathematics as a separate qualification and which can be taught alongside AS or A level mathematics, is a very useful qualification in its own right. It broadens and reinforces the content of AS and A level mathematics, introduces complex numbers and matrices, and gives students the opportunity to extend their knowledge in applied mathematics and logical reasoning. This breadth and depth of study is very valuable for supporting the transition to degree level work and employment in mathematical disciplines.

Aims and objectives

5. AS and A level specifications in further mathematics must encourage students to:

- understand mathematics and mathematical processes in ways that promote confidence, foster enjoyment and provide a strong foundation for progress to further study

- extend their range of mathematical skills and techniques
- understand coherence and progression in mathematics and how different areas of mathematics are connected
- apply mathematics in other fields of study and be aware of the relevance of mathematics to the world of work and to situations in society in general
- use their mathematical knowledge to make logical and reasoned decisions in solving problems both within pure mathematics and in a variety of contexts, and communicate the mathematical rationale for these decisions clearly
- reason logically and recognise incorrect reasoning
- generalise mathematically
- construct mathematical proofs
- use their mathematical skills and techniques to solve challenging problems which require them to decide on the solution strategy
- recognise when mathematics can be used to analyse and solve a problem in context
- represent situations mathematically and understand the relationship between problems in context and mathematical models that may be applied to solve them
- draw diagrams and sketch graphs to help explore mathematical situations and interpret solutions
- make deductions and inferences and draw conclusions by using mathematical reasoning
- interpret solutions and communicate their interpretation effectively in the context of the problem
- read and comprehend mathematical arguments, including justifications of methods and formulae, and communicate their understanding
- read and comprehend articles concerning applications of mathematics and communicate their understanding
- use technology such as calculators and computers effectively, and recognise when such use may be inappropriate
- take increasing responsibility for their own learning and the evaluation of their own mathematical development

Subject content

Structure

6. A level further mathematics has a prescribed core which must comprise approximately 50% of its content. The core content is set out in sections A to I. For the remaining 50% of the content, different options are available. The content of these options is not prescribed and will be defined within the different awarding organisations' specifications; these options could build from the applied content in A level Mathematics, they could introduce new applications, or they could extend further the core content defined below, or they could involve some combination of these. Any optional content must be at the same level of demand as the prescribed core.

7. In any AS further mathematics specification, at least one route must be available to allow the qualification to be taught alongside AS mathematics: the content of the components that make up this route may either be new, or may build on the content of AS mathematics, but must not significantly overlap with or depend upon other A level mathematics content.

8. At least 30% (approximately) of the content of any AS further mathematics specification must be taken from the prescribed core content of A level further mathematics. Some of this is prescribed and some is to be selected by the awarding organisation, as follows:

- core content that must be included in any AS further mathematics specification is indicated in sections B to D below using bold text within square brackets. This content must represent approximately 20% of the overall content of AS further mathematics
- awarding organisations must select other content from the non-bold statements in the prescribed core content of A level further mathematics to be in their AS further mathematics specifications; this should represent a minimum of 10% (approximately) of the AS further mathematics content

Background knowledge

9. AS and A level further mathematics specifications must build on the skills, knowledge and understanding set out in the whole GCSE subject content for mathematics and the subject content for AS and A level mathematics. Problem solving, proof and mathematical modelling will be assessed in further mathematics in the context of the wider knowledge which students taking AS/A level further mathematics will have studied. The required knowledge and skills common to all AS further mathematics specifications are shown in the following tables in bold text within square brackets. Occasionally knowledge and skills from the content of A level

mathematics which is not in AS mathematics are assumed; this is indicated in brackets in the relevant content statements.

Overarching themes

10. A level specifications in further mathematics must require students to demonstrate the following overarching knowledge and skills. These must be applied, along with associated mathematical thinking and understanding, across the whole of the detailed content set out below. The knowledge and skills are similar to those specified for A level mathematics but they will be examined against further mathematics content and contexts.

OT1 Mathematical argument, language and proof

AS and A level further mathematics specifications must use the mathematical notation set out in appendix A and must require students to recall the mathematical formulae and identities set out in appendix B.

	Knowledge/Skill
OT1.1	**[Construct and present mathematical arguments through appropriate use of diagrams; sketching graphs; logical deduction; precise statements involving correct use of symbols and connecting language, including: constant, coefficient, expression, equation, function, identity, index, term, variable]**
OT1.2	**[Understand and use mathematical language and syntax as set out in the glossary]**
OT1.3	**[Understand and use language and symbols associated with set theory, as set out in the glossary]**
OT1.4	Understand and use the definition of a function; domain and range of functions
OT1.5	**[Comprehend and critique mathematical arguments, proofs and justifications of methods and formulae, including those relating to applications of mathematics]**

OT2 Mathematical problem solving

	Knowledge/Skill
OT2.1	**[Recognise the underlying mathematical structure in a situation and simplify and abstract appropriately to enable problems to be solved]**
OT2.2	**[Construct extended arguments to solve problems presented in an unstructured form, including problems in context]**
OT2.3	**[Interpret and communicate solutions in the context of the original problem]**

OT2.6	**[Understand the concept of a mathematical problem solving cycle, including specifying the problem, collecting information, processing and representing information and interpreting results, which may identify the need to repeat the cycle]**
OT2.7	**[Understand, interpret and extract information from diagrams and construct mathematical diagrams to solve problems]**

OT3 Mathematical modelling

	Knowledge/Skill
OT3.1	**[Translate a situation in context into a mathematical model, making simplifying assumptions]**
OT3.2	**[Use a mathematical model with suitable inputs to engage with and explore situations (for a given model or a model constructed or selected by the student)]**
OT3.3	**[Interpret the outputs of a mathematical model in the context of the original situation (for a given model or a model constructed or selected by the student)]**
OT3.4	**[Understand that a mathematical model can be refined by considering its outputs and simplifying assumptions; evaluate whether the model is appropriate]**
OT3.5	**[Understand and use modelling assumptions]**

Use of technology

11. The use of technology, in particular mathematical graphing tools and spreadsheets, must permeate the study of AS and A level further mathematics. Calculators used must include the following features:

- an iterative function
- the ability to perform calculations with matrices up to at least order 3 x 3
- the ability to compute summary statistics and access probabilities from standard statistical distributions

Detailed content statements

12. A level specifications in further mathematics must include the following content. This, assessed in the context of the overarching themes, makes up approximately 50% of the total content of the A level.

A Proof

	Content
A1	Construct proofs using mathematical induction; contexts include sums of series, divisibility, and powers of matrices

B Complex numbers

	Content
B1	**[Solve any quadratic equation with real coefficients; solve cubic or quartic equations with real coefficients (given sufficient information to deduce at least one root for cubics or at least one complex root or quadratic factor for quartics)]**
B2	**[Add, subtract, multiply and divide complex numbers in the form $x + \mathrm{i}y$ with x and y real; understand and use the terms 'real part' and 'imaginary part']**
B3	**[Understand and use the complex conjugate; know that non-real roots of polynomial equations with real coefficients occur in conjugate pairs]**
B4	**[Use and interpret Argand diagrams]**
B5	**[Convert between the Cartesian form and the modulus-argument form of a complex number (knowledge of radians is assumed)]**
B6	**[Multiply and divide complex numbers in modulus-argument form (knowledge of radians and compound angle formulae is assumed)]**
B7	**[Construct and interpret simple loci in the Argand diagram such as $\lvert z-a \rvert > r$ and $\arg(z-a) = \theta$ (knowledge of radians is assumed)]**
B8	Understand de Moivre's theorem and use it to find multiple angle formulae and sums of series
B9	Know and use the definition $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ and the form $z = r\mathrm{e}^{\mathrm{i}\theta}$
B10	Find the *n* distinct n^{th} roots of $r\mathrm{e}^{\mathrm{i}\theta}$ for $r \neq 0$ and know that they form the vertices of a regular *n*-gon in the Argand diagram
B11	Use complex roots of unity to solve geometric problems

13. For section C students must demonstrate the ability to use calculator technology that will enable them to perform calculations with matrices up to at least order 3 x 3.

C Matrices

	Content
C1	**[Add, subtract and multiply conformable matrices; multiply a matrix by a scalar]**
C2	**[Understand and use zero and identity matrices]**

C3	**[Use matrices to represent linear transformations in 2-D; successive transformations; single transformations in 3-D (3-D transformations confined to reflection in one of $x = 0$, $y = 0$, $z = 0$ or rotation about one of the coordinate axes) (knowledge of 3-D vectors is assumed)]**
C4	**[Find invariant points and lines for a linear transformation]**
C5	**[Calculate determinants of 2 x 2]** and 3 x 3 matrices and interpret as scale factors, including the effect on orientation
C6	**[Understand and use singular and non-singular matrices; properties of inverse matrices]** **[Calculate and use the inverse of non-singular 2 x 2 matrices]** and 3 x 3 matrices
C7	Solve three linear simultaneous equations in three variables by use of the inverse matrix
C8	Interpret geometrically the solution and failure of solution of three simultaneous linear equations

D Further algebra and functions

	Content
D1	**[Understand and use the relationship between roots and coefficients of polynomial equations up to quartic equations]**
D2	**[Form a polynomial equation whose roots are a linear transformation of the roots of a given polynomial equation (of at least cubic degree)]**
D3	Understand and use formulae for the sums of integers, squares and cubes and use these to sum other series
D4	Understand and use the method of differences for summation of series including use of partial fractions
D5	Find the Maclaurin series of a function including the general term
D6	Recognise and use the Maclaurin series for e^x, $\ln(1+x)$, $\sin x$, $\cos x$ and $(1+x)^n$, and be aware of the range of values of x for which they are valid (proof not required)

E Further calculus

	Content
E1	Evaluate improper integrals where either the integrand is undefined at a value in the range of integration or the range of integration extends to infinity
E2	Derive formulae for and calculate volumes of revolution
E3	Understand and evaluate the mean value of a function
E4	Integrate using partial fractions (extend to quadratic factors $ax^2 + c$ in the denominator)

E5	Differentiate inverse trigonometric functions
E6	Integrate functions of the form $\left(a^2-x^2\right)^{-\frac{1}{2}}$ and $\left(a^2+x^2\right)^{-1}$ and be able to choose trigonometric substitutions to integrate associated functions

F Further vectors

	Content
F1	Understand and use the vector and Cartesian forms of an equation of a straight line in 3D
F2	Understand and use the vector and Cartesian forms of the equation of a plane
F3	Calculate the scalar product and use it to express the equation of a plane, and to calculate the angle between two lines, the angle between two planes and the angle between a line and a plane
F4	Check whether vectors are perpendicular by using the scalar product
F5	Find the intersection of a line and a plane Calculate the perpendicular distance between two lines, from a point to a line and from a point to a plane

G Polar coordinates

	Content
G1	Understand and use polar coordinates and be able to convert between polar and cartesian coordinates
G2	Sketch curves with r given as a function of θ, including use of trigonometric functions
G3	Find the area enclosed by a polar curve

H Hyperbolic functions

	Content
H1	Understand the definitions of hyperbolic functions $\sinh x$, $\cosh x$ and $\tanh x$, including their domains and ranges, and be able to sketch their graphs
H2	Differentiate and integrate hyperbolic functions
H3	Understand and be able to use the definitions of the inverse hyperbolic functions and their domains and ranges
H4	Derive and use the logarithmic forms of the inverse hyperbolic functions

H5	Integrate functions of the form $\left(x^2+a^2\right)^{-\frac{1}{2}}$ and $\left(x^2-a^2\right)^{-\frac{1}{2}}$ and be able to choose substitutions to integrate associated functions

I Differential equations

	Content
I1	Find and use an integrating factor to solve differential equations of form $\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{P}(x)y=\mathrm{Q}(x)$ and recognise when it is appropriate to do so
I2	Find both general and particular solutions to differential equations
I3	Use differential equations in modelling in kinematics and in other contexts
I4	Solve differential equations of form $y''+ay'+by=0$ where a and b are constants by using the auxiliary equation
I5	Solve differential equations of form $y''+ay'+by=\mathrm{f}(x)$ where a and b are constants by solving the homogeneous case and adding a particular integral to the complementary function (in cases where $\mathrm{f}(x)$ is a polynomial, exponential or trigonometric function)
I6	Understand and use the relationship between the cases when the discriminant of the auxiliary equation is positive, zero and negative and the form of solution of the differential equation
I7	Solve the equation for simple harmonic motion $\ddot{x}=-\omega^2 x$ and relate the solution to the motion
I8	Model damped oscillations using 2^{nd} order differential equations and interpret their solutions
I9	Analyse and interpret models of situations with one independent variable and two dependent variables as a pair of coupled 1^{st} order simultaneous equations and be able to solve them, for example predator-prey models

Appendix A: mathematical notation for AS qualifications and A levels in mathematics and further mathematics

The tables below set out the notation that must be used by AS and A level mathematics and further mathematics specifications. Students will be expected to understand this notation without need for further explanation.

Mathematics students will not be expected to understand notation that relates only to further mathematics content. Further mathematics students will be expected to understand all notation in the list.

For further mathematics, the notation for the core content is listed under sub headings indicating 'further mathematics only'. In this subject, awarding organisations are required to include, in their specifications, content that is additional to the core content. They will therefore need to add to the notation list accordingly.

AS students will be expected to understand notation that relates to AS content, and will not be expected to understand notation that relates only to A level content.

1	**Set Notation**	
1.1	$\in$	is an element of
1.2	$\notin$	is not an element of
1.3	$\subseteq$	is a subset of
1.4	$\subset$	is a proper subset of
1.5	$\{x_1, x_2, \ldots\}$	the set with elements $x_1, x_2, \ldots$
1.6	$\{x: \ldots\}$	the set of all x such that $\ldots$
1.7	$\mathrm{n}(A)$	the number of elements in set A
1.8	$\varnothing$	the empty set
1.9	$\mathcal{E}$	the universal set
1.10	A'	the complement of the set A
1.11	$\mathbb{N}$	the set of natural numbers, $\{1, 2, 3, \ldots\}$
1.12	$\mathbb{Z}$	the set of integers, $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$
1.13	$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, \ldots\}$
1.14	$\mathbb{Z}_0^+$	the set of non-negative integers, $\{0, 1, 2, 3, \ldots\}$
1.15	$\mathbb{R}$	the set of real numbers

1.16	$\mathbb{Q}$	the set of rational numbers, $\left\{\frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{Z}^{+}\right\}$
1.17	$\cup$	union
1.18	$\cap$	intersection
1.19	$(x,\ y)$	the ordered pair x, y
1.20	$[a,\ b]$	the closed interval $\{x \in \mathbb{R} : a \le x \le b\}$
1.21	$[a,\ b)$	the interval $\{x \in \mathbb{R} : a \le x < b\}$
1.22	$(a,\ b]$	the interval $\{x \in \mathbb{R} : a < x \le b\}$
1.23	$(a,\ b)$	the open interval $\{x \in \mathbb{R} : a < x < b\}$
1	**Set Notation (Further Mathematics only)**	
1.24	$\mathbb{C}$	the set of complex numbers
2	**Miscellaneous Symbols**	
2.1	$=$	is equal to
2.2	$\neq$	is not equal to
2.3	$\equiv$	is identical to or is congruent to
2.4	$\approx$	is approximately equal to
2.5	∞	infinity
2.6	$\propto$	is proportional to
2.7	$\therefore$	therefore
2.8	$\because$	because
2.9	$<$	is less than
2.10	$\leqslant$, $\le$	is less than or equal to, is not greater than
2.11	$>$	is greater than
2.12	$\geqslant$, $\ge$	is greater than or equal to, is not less than
2.13	$p \Rightarrow q$	p implies q (if p then q)
2.14	$p \Leftarrow q$	p is implied by q (if q then p)
2.15	$p \Leftrightarrow q$	p implies and is implied by q (p is equivalent to q)
2.16	a	first term for an arithmetic or geometric sequence
2.17	l	last term for an arithmetic sequence
2.18	d	common difference for an arithmetic sequence
2.19	r	common ratio for a geometric sequence
2.20	S_n	sum to n terms of a sequence

2.21	S_∞	sum to infinity of a sequence
3	**Operations**	
3.1	$a+b$	a plus b
3.2	$a-b$	a minus b
3.3	$a\times b,\ ab,\ a.b$	a multiplied by b
3.4	$a \div b,\ \frac{a}{b}$	a divided by b
3.5	$\sum_{i=1}^{n} a_i$	$a_1 + a_2 + \ldots + a_n$
3.6	$\prod_{i=1}^{n} a_i$	$a_1 \times a_2 \times \ldots \times a_n$
3.7	$\sqrt{a}$	the non-negative square root of a
3.8	$\lvert a \rvert$	the modulus of a
3.9	$n!$	n factorial: $n! = n\times(n-1)\times...\times 2\times 1,\ n\in\mathbb{N};\ 0!=1$
3.10	$\binom{n}{r},\ {}^nC_r,\ {}_nC_r$	the binomial coefficient $\frac{n!}{r!(n-r)!}$ for $n, r \in \mathbb{Z}_0^+, r \leqslant n$ or $\frac{n(n-1)\ldots(n-r+1)}{r!}$ for $n \in \mathbb{Q}, r \in \mathbb{Z}_0^+$
4	**Functions**	
4.1	$\mathrm{f}(x)$	the value of the function f at x
4.2	$\mathrm{f}: x \mapsto y$	the function f maps the element x to the element y
4.3	f^{-1}	the inverse function of the function f
4.4	gf	the composite function of f and g which is defined by $\mathrm{gf}(x) = \mathrm{g}(\mathrm{f}(x))$
4.5	$\lim_{x\to a} \mathrm{f}(x)$	the limit of $\mathrm{f}(x)$ as x tends to a
4.6	$\Delta x,\ \delta x$	an increment of x
4.7	$\frac{\mathrm{d}y}{\mathrm{d}x}$	the derivative of y with respect to x
4.8	$\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$	the nth derivative of y with respect to x
4.9	$\mathrm{f}'(x),\ \mathrm{f}''(x),\ \ldots,\ \mathrm{f}^{(n)}(x)$	the first, second, ..., n^{th} derivatives of $\mathrm{f}(x)$ with respect to x
4.10	$\dot{x},\ \ddot{x},\ \ldots$	the first, second, ... derivatives of x with respect to t

4.11	$\int y \, dx$	the indefinite integral of y with respect to x
4.12	$\int_a^b y \, dx$	the definite integral of y with respect to x between the limits $x = a$ and $x = b$
5	**Exponential and Logarithmic Functions**	
5.1	e	base of natural logarithms
5.2	e^x, $\exp x$	exponential function of x
5.3	$\log_a x$	logarithm to the base a of x
5.4	$\ln x$, $\log_e x$	natural logarithm of x
6	**Trigonometric Functions**	
6.1	$\left.\begin{array}{l}\sin, \cos, \tan, \\ \operatorname{cosec}, \sec, \cot\end{array}\right\}$	the trigonometric functions
6.2	$\left.\begin{array}{l}\sin^{-1}, \cos^{-1}, \tan^{-1}, \\ \arcsin, \arccos, \arctan\end{array}\right\}$	the inverse trigonometric functions
6.3	$^\circ$	degrees
6.4	rad	radians
6	**Trigonometric and Hyperbolic Functions (Further Mathematics only)**	
6.5	$\left.\begin{array}{l}\operatorname{cosec}^{-1}, \sec^{-1}, \cot^{-1}, \\ \operatorname{arccosec}, \operatorname{arcsec}, \operatorname{arccot}\end{array}\right\}$	the inverse trigonometric functions
6.6	$\left.\begin{array}{l}\sinh, \cosh, \tanh, \\ \operatorname{cosech}, \operatorname{sech}, \coth\end{array}\right\}$	the hyperbolic functions
6.7	$\left.\begin{array}{l}\sinh^{-1}, \cosh^{-1}, \tanh^{-1}, \\ \operatorname{cosech}^{-1}, \operatorname{sech}^{-1}, \coth^{-1}\end{array}\right\}$ $\left.\begin{array}{l}\operatorname{arsinh}, \operatorname{arcosh}, \operatorname{artanh}, \\ \operatorname{arcosech}, \operatorname{arsech}, \operatorname{arcoth}\end{array}\right\}$	the inverse hyperbolic functions
7	**Complex Numbers (Further Mathematics only)**	
7.1	i, j	square root of -1
7.2	$x + iy$	complex number with real part x and imaginary part y
7.3	$r(\cos\theta + i\sin\theta)$	modulus argument form of a complex number with modulus r and argument θ
7.4	z	a complex number, $z = x + iy = r(\cos\theta + i\sin\theta)$
7.5	$\operatorname{Re}(z)$	the real part of z, $\operatorname{Re}(z) = x$
7.6	$\operatorname{Im}(z)$	the imaginary part of z, $\operatorname{Im}(z) = y$

7.7	$\lvert z \rvert$	the modulus of z, $\lvert z \rvert = \sqrt{x^2 + y^2}$
7.8	$\arg(z)$	the argument of z, $\arg(z) = \theta,\ -\pi < \theta \le \pi$
7.9	z^*	the complex conjugate of z, $x - \mathrm{i}y$
8	**Matrices (Further Mathematics only)**	
8.1	$\mathbf{M}$	a matrix $\mathbf{M}$
8.2	$\mathbf{0}$	zero matrix
8.3	$\mathbf{I}$	identity matrix
8.4	$\mathbf{M}^{-1}$	the inverse of the matrix $\mathbf{M}$
8.5	$\mathbf{M}^{\mathrm{T}}$	the transpose of the matrix $\mathbf{M}$
8.6	Δ, det $\mathbf{M}$ or $\lvert \mathbf{M} \rvert$	the determinant of the square matrix $\mathbf{M}$
8.7	$\mathbf{Mr}$	Image of column vector $\mathbf{r}$ under the transformation associated with the matrix $\mathbf{M}$
9	**Vectors**	
9.1	$\mathbf{a}$, $\underline{a}$, $\utilde{a}$	the vector $\mathbf{a}$, $\underline{a}$, $\utilde{a}$; these alternatives apply throughout section 9
9.2	$\overrightarrow{\mathrm{AB}}$	the vector represented in magnitude and direction by the directed line segment AB
9.3	$\hat{\mathbf{a}}$	a unit vector in the direction of $\mathbf{a}$
9.4	$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$	unit vectors in the directions of the cartesian coordinate axes
9.5	$\lvert \mathbf{a} \rvert$, a	the magnitude of $\mathbf{a}$
9.6	$\lvert \overrightarrow{\mathrm{AB}} \rvert$, AB	the magnitude of $\overrightarrow{\mathrm{AB}}$
9.7	$\begin{pmatrix} a \\ b \end{pmatrix}$, $a\mathbf{i} + b\mathbf{j}$	column vector and corresponding unit vector notation
9.8	$\mathbf{r}$	position vector
9.9	$\mathbf{s}$	displacement vector
9.10	$\mathbf{v}$	velocity vector
9.11	$\mathbf{a}$	acceleration vector

9	**Vectors (Further Mathematics only)**	
9.12	**a.b**	the scalar product of **a** and **b**
10	**Differential Equations (Further Mathematics only)**	
10.1	ω	angular speed
11	**Probability and Statistics**	
11.1	$A, B, C,$ etc.	events
11.2	$A \cup B$	union of the events A and B
11.3	$A \cap B$	intersection of the events A and B
11.4	$\mathrm{P}(A)$	probability of the event A
11.5	A'	complement of the event A
11.6	$\mathrm{P}(A \mid B)$	probability of the event A conditional on the event B
11.7	$X, Y, R,$ etc.	random variables
11.8	$x, y, r,$ etc.	values of the random variables X, Y, R etc.
11.9	$x_1, x_2, \ldots$	values of observations
11.10	$f_1, f_2, \ldots$	frequencies with which the observations $x_1, x_2, \ldots$ occur
11.11	$\mathrm{p}(x), \mathrm{P}(X = x)$	probability function of the discrete random variable X
11.12	$p_1, p_2, \ldots$	probabilities of the values $x_1, x_2, \ldots$ of the discrete random variable X
11.13	$\mathrm{E}(X)$	expectation of the random variable X
11.14	$\mathrm{Var}(X)$	variance of the random variable X
11.15	$\sim$	has the distribution
11.16	$\mathrm{B}(n, p)$	binomial distribution with parameters n and p, where n is the number of trials and p is the probability of success in a trial
11.17	q	$q = 1 - p$ for binomial distribution
11.18	$\mathrm{N}(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2
11.19	$Z \sim \mathrm{N}(0,1)$	standard Normal distribution
11.20	ϕ	probability density function of the standardised Normal variable with distribution $\mathrm{N}(0, 1)$
11.21	Φ	corresponding cumulative distribution function
11.22	μ	population mean
11.23	σ^2	population variance
11.24	σ	population standard deviation

11.25	$\bar{x}$	sample mean
11.26	s^2	sample variance
11.27	s	sample standard deviation
11.28	H_0	Null hypothesis
11.29	H_1	Alternative hypothesis
11.30	r	product moment correlation coefficient for a sample
11.31	ρ	product moment correlation coefficient for a population
12	**Mechanics**	
12.1	kg	kilograms
12.2	m	metres
12.3	km	kilometres
12.4	m/s, m s^{-1}	metres per second (velocity)
12.5	m/s^2, m s^{-2}	metres per second per second (acceleration)
12.6	F	Force or resultant force
12.7	N	Newton
12.8	N m	Newton metre (moment of a force)
12.9	t	time
12.10	s	displacement
12.11	u	initial velocity
12.12	v	velocity or final velocity
12.13	a	acceleration
12.14	g	acceleration due to gravity
12.15	μ	coefficient of friction

Appendix B: mathematical formulae and identities

Students must be able to use the following formulae and identities for AS and A level further mathematics, without these formulae and identities being provided, either in these forms or in equivalent forms. These formulae and identities may only be provided where they are the starting point for a proof or as a result to be proved.

Pure Mathematics

Quadratic Equations

$ax^2 + bx + c = 0$ has roots $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Laws of Indices

$a^x a^y \equiv a^{x+y}$

$a^x \div a^y \equiv a^{x-y}$

$(a^x)^y \equiv a^{xy}$

Laws of Logarithms

$x = a^n \Leftrightarrow n = \log_a x$ for $a > 0$ and $x > 0$

$\log_a x + \log_a y \equiv \log_a (xy)$

$\log_a x - \log_a y \equiv \log_a \left(\dfrac{x}{y}\right)$

$k \log_a x \equiv \log_a \left(x^k\right)$

Coordinate Geometry

A straight line graph, gradient m passing through $\left(x_1, y_1\right)$ has equation

$y - y_1 = m\left(x - x_1\right)$

Straight lines with gradients m_1 and m_2 are perpendicular when $m_1 m_2 = -1$

Sequences

General term of an arithmetic progression:

$u_n = a + (n-1)d$

General term of a geometric progression:

$u_n = ar^{n-1}$

Trigonometry

In the triangle ABC

Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$

Area $= \dfrac{1}{2}ab\sin C$

$\cos^2 A + \sin^2 A \equiv 1$
$\sec^2 A \equiv 1 + \tan^2 A$
$\operatorname{cosec}^2 A \equiv 1 + \cot^2 A$

$\sin 2A \equiv 2\sin A\cos A$
$\cos 2A \equiv \cos^2 A - \sin^2 A$

$\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$

Mensuration

Circumference and Area of circle, radius *r* and diameter *d*:

$C = 2\pi r = \pi d \qquad A = \pi r^2$

Pythagoras' Theorem: In any right-angled triangle where *a*, *b* and *c* are the lengths of the sides and *c* is the hypotenuse:

$c^2 = a^2 + b^2$

Area of a trapezium = $\dfrac{1}{2}(a+b)h$, where *a* and *b* are the lengths of the parallel sides and *h* is their perpendicular separation.

Volume of a prism = area of cross section × length

For a circle of radius r, where an angle at the centre of θ radians subtends an arc of length s and encloses an associated sector of area A:

$$s = r\theta \quad A = \frac{1}{2}r^2\theta$$

Complex Numbers

For two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Loci in the Argand diagram:

$\lvert z-a \rvert = r$	is a circle radius r centred at a
$\arg(z-a) = \theta$	is a half line drawn from a at angle θ to a line parallel to the positive real axis

Exponential Form:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Matrices

For a 2 by 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the determinant $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

the inverse is $\frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

The transformation represented by matrix **AB** is the transformation represented by matrix **B** followed by the transformation represented by matrix **A**.

For matrices **A**, **B**:

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Algebra

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

For $ax^2+bx+c=0$ with roots α and β:

$$\alpha+\beta = \frac{-b}{a} \qquad \alpha\beta = \frac{c}{a}$$

For $ax^3+bx^2+cx+d=0$ with roots α, β and γ:

$$\sum\alpha = \frac{-b}{a} \qquad \sum\alpha\beta = \frac{c}{a} \qquad \alpha\beta\gamma = \frac{-d}{a}$$

Hyperbolic Functions

$$\cosh x \equiv \frac{1}{2}\left(e^{x}+e^{-x}\right)$$

$$\sinh x \equiv \frac{1}{2}\left(e^{x}-e^{-x}\right)$$

$$\tanh x \equiv \frac{\sinh x}{\cosh x}$$

Calculus and Differential Equations

<u>Differentiation</u>

Function	Derivative
x^n	nx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
$\sinh kx$	$k\cosh kx$
$\cosh kx$	$k\sinh kx$
e^{kx}	ke^{kx}
$\ln x$	$\frac{1}{x}$
$f(x)+g(x)$	$f'(x)+g'(x)$
$f(x)g(x)$	$f'(x)g(x)+f(x)g'(x)$
$f(g(x))$	$f'(g(x))g'(x)$

Integration

Function	Integral
x^n	$\frac{1}{n+1}x^{n+1}+c,\ n \neq -1$
$\cos kx$	$\frac{1}{k}\sin kx + c$
$\sin kx$	$-\frac{1}{k}\cos kx + c$
$\cosh kx$	$\frac{1}{k}\sinh kx + c$
$\sinh kx$	$\frac{1}{k}\cosh kx + c$
e^{kx}	$\frac{1}{k}\mathrm{e}^{kx} + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c,\ x \neq 0$
$\mathrm{f}'(x)+\mathrm{g}'(x)$	$\mathrm{f}(x)+\mathrm{g}(x)+c$
$\mathrm{f}'(\mathrm{g}(x))\mathrm{g}'(x)$	$\mathrm{f}(\mathrm{g}(x))+c$

Area under a curve $= \int_a^b y \ \mathrm{d}x \ \ (y \geq 0)$

Volumes of revolution about the x and y axes:

$$V_x = \pi\int_a^b y^2 dx \qquad V_y = \pi\int_c^d x^2 dy$$

Simple Harmonic Motion:

$$\ddot{x} = -\omega^2 x$$

Vectors

$|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \sqrt{(x^2 + y^2 + z^2)}$

Scalar product of two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where θ is the acute angle between the vectors $\mathbf{a}$ and $\mathbf{b}$

The equation of the line through the point with position vector $\mathbf{a}$ parallel to vector $\mathbf{b}$ is:

$$\mathbf{r} = \mathbf{a} + t\mathbf{b}$$

The equation of the plane containing the point with position vector $\mathbf{a}$ and perpendicular to vector $\mathbf{n}$ is:

$$(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$$

Mechanics

Forces and Equilibrium

Weight = mass $\times g$

Friction: $F \leq \mu R$

Newton's second law in the form: $F = ma$

Kinematics

For motion in a straight line with variable acceleration:

$$v = \frac{\mathrm{d}r}{\mathrm{d}t} \qquad a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2}$$

$$r = \int v \, \mathrm{d}t \qquad v = \int a \, \mathrm{d}t$$

Statistics

The mean of a set of data: $\overline{x} = \frac{\sum x}{n} = \frac{\sum fx}{\sum f}$

The standard Normal variable: $Z = \frac{X - \mu}{\sigma}$ where $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$

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